

Regularity results for some models in geophysical fluid dynamics

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A thesis submitted for the degree of
Doctor of Mathematics at the
Universidad Autónoma de Madrid

Supervised by:
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Madrid, 2019

*A Tata,
porque tu voz sigue sonando en mi corazón.*

Agradecimientos

Como cantaba el salsero Hector Lavoe: *todo tiene su final, nada dura para siempre*. Y así es, después de estos cuatro años se termina esta etapa, llena de muy buenos (y no tan buenos) momentos. Un carrusel de emociones que sin duda no ha descarrilado gracias al apoyo constante de mi familia y mis amigos.

En primer lugar, quiero agradecer a mi director de tesis Antonio Córdoba por su inagotable ayuda, paciencia y sabios consejos que me ha dado durante estos años. Más allá de ser un matemático excelente y un amante de la belleza, ha sido un guía excepcional en este camino lleno de obstáculos.

Antes de continuar también me gustaría mencionar la inestimable ayuda de grandísimos matemáticos y amigos con los que he podido colaborar en distintos proyectos en estos años: Ángel David, Antonio Córdoba, Aythami Bethencourt, So Takao, Fernando Chamizo y Albert Mas. I'm also indebted to Vlad Vicol for his infinite wisdom, hospitality and kindness when I was visiting the Courant Institute. También gracias a Juan José Velázquez por darme la oportunidad de ir a Bonn, aprender de ti y trabajar contigo.

Pero no adelantemos más acontecimientos, comencemos por el principio:

En septiembre de 2013 llegué a Madrid para hacer el máster de Matemáticas y Aplicaciones en la Universidad Autónoma. Desde un primer momento, pude disfrutar de un despacho en el ICMAT y asistir a todos los seminarios (algunos con pizza), congresos y actividades matemáticas que se hacían en el instituto. Sin duda, esto fue posible gracias a Diego Córdoba, que me introdujo en el mundo de la investigación y en particular de los fluidos.

Al mismo tiempo (juraría que el mismo día), llegó al ICMAT un nuevo estudiante de doctorado de Antonio, Eric Latorre. Por casualidades de la vida (en realidad gracias a Eduardo Frechilla), coincidimos juntos en el despacho 308 (léase *three-oh-eight*). Conectamos a la perfección y en muy poco tiempo pasó a ser mi *buddy*. Entre artículos y teoremas, siempre hubo tiempo para charlas, conciertos musicales, clases de ukelele, karaokes, bailes...Sin olvidar el squash, donde Eric fue mi auténtico *sensei*. Gracias por esos momentos, por tu amistad y hacer que las pequeñas frustraciones (ϵ - δ que no cuadraban) se disiparan con tu compañía. También agradecer su simpatía a los otros habitantes del despacho: Alberto Navarro (gran jugador de LOL), Adrián

González (forofo de la teína y las cosas que no conmutan) y Diego González (buen jardinero y salsero cubano).

Durante el máster y principio del doctorado también tuve la suerte de conocer a muchas personas que fueron importantes para mí. Gracias a Alberto Martín por la ayuda altruista que me ofreciste desde un primer momento con todas mis dudas (para tí, triviales). También estoy en deuda con Dani Lear, que siempre me acogió con agrado en mis constantes visitas a su despacho, ya fuese para desahogarme o para charlar de cualquier cosa. No me puedo olvidar de Ángel Castro, que a pesar de su marcado *carácter madrileño*, siempre estuvo dispuesto a echarme una mano cuando lo necesitaba. *Merci a mon ami monsieur* Lazar, con el que pasé horas y horas en la pizarra buscando *la cancelación* y jugando largas partidas de ajedrez. Gracias también a Juanjo, Víctor y Jorge porque nunca se quejaron cuando fui a verlos en busca de una pausa, un respiro o un consejo. A mi canaria Jezabel, por los bailes de los jueves y cenitas *sanas* previas en el McDonalds. A los *gemeliers*, Manuel Jesús \cong José Luis por las pachangas de fútbol, y a Cristina que nos lió para salir en la tele.

Fugaz y sin remordimiento pasó el tiempo, *tempus fugit*, pero los teoremas no salían. En ese momento de miedo e inseguridad, tus peores demonios salen y la frustración te come. De manera casual y natural, empecé a tener mas contacto y cercanía con Ángel David. Con su risa llamativa, su reloj de bolsillo y colección \aleph_0 de libros, Ángel era todo un grande (en el amplio sentido de la palabra). Sin *querer queriendo*, comenzamos a colaborar juntos (co-autor y co-comensal). Mas allá de las horas de trabajo y de la sorprendente capacidad de ideas por hora que tenía (algunas funcionaban), descubrí a una gran persona y un fiel amigo. Gracias por estar siempre ahí, sobre todo en los momentos *quánticamente* complicados.

María de los Ángeles, *la Mari* para los amigos, fue como un soplo de aire fresco. Una persona de bondad infinita y un corazón enorme. Gracias por ser tan buena y mantener un halo de paz en el grupo. Gracias a Carlos Pastor por tener siempre una respuesta para cualquier cosa y disfrutar tanto de las cenas navideñas en casa.

Muchas otras personas también han sido un pilar fundamental durante este tiempo. Mis amigos de la universidad que me ayudaron a centrarme en algunos momentos de total dispersión y a seguir por el *buen camino*. Gracias a Marta (abre los ojos) perseverante y cariñosa, Zaida (bufanda al cuello) divertida y risueña, Marcos (Don Apuntes) espiritual y vacilón, y Aythami (bomba de humo) inteligente y honrado. Este último merece una mención especial, pues a parte de ser un gran amigo, hemos compartido muchos momentos matemáticos (y algunos teoremas) este último año. También a Yoni y Sary que me cuidaron, incluso a veces me consintieron, desde el día que nos conocimos. Gracias a Héctor, con el que me crié y quiero como a un hermano, porque sé que puedo contar contigo siempre que quiera.

Por último y más importante, a mi familia. Estoy completamente en deuda con mis tíos y mis primas, Berto, Chiky, Laura y Marta, quienes me acogieron y me hicieron sentir como en casa desde el primer día. El cariño y afecto que me han dado, ha sido vital para que esta tesis saliera adelante y por ello, parte de ella también es vuestra. Todos estos recuerdos siempre estarán conmigo.

A mis padres, Sara y Miguel Ángel, que me convirtieron en la persona que soy, y son para mí un modelo a seguir en la vida. Gracias mami por transmitirme tu positividad y energía, por nunca ponerte barreras y siempre superarte. Gracias papi por enseñarme a valorar las cosas y hacer que diera lo mejor de mí. Gracias a los dos por *invertir* en mí y sacrificarse para darme lo mejor. A mi hermana Amelia, a quien le debo tanto, siempre animándome con su sonrisa permanente y contagiosa. También a una reciente incorporación, Jorge Hernández, una auténtica bestia del guión y del Trivial Pursuit. A mi *primus* Jorge, el primo *Zumosol* que todo niño quiere tener. Gracias por pasar tanto tiempo conmigo y protegerme.

Todo pasa y todo acaba, y así terminan estos agradecimientos, que se quedan cortos a la hora de transmitir todo lo que siento. Volviendo a repasar mentalmente todos los recuerdos de estos cuatro años, me quedo sin dudarlo con uno: el momento en el que me crucé contigo en el camino. Tania, gracias por compartir tu vida conmigo y estar presente día a día con tu apoyo incondicional. Tenerte a mi lado hace todo más fácil y sencillo. Te quiero, hoy y siempre.

Diego Alonso Orán,
Madrid, Febrero 2019.

This thesis has been funded by a Severo Ochoa FPI scholarship for Centres of Excellence in R&D (SEV-2015-0554) and by the grant MTM2017-83496-P from the Spanish Ministry of Economy and Competitiveness.

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ICMAT-UAM

February 11, 2019

Abstract

This thesis centers on the study of two different problems of partial differential equations arising from geophysics and fluid mechanics: the surface quasi-geostrophic equation and the so called, Incompressible Slice Model.

The surface quasi-geostrophic equation is a two dimensional nonlocal partial differential equation of geophysical importance, describing the evolution of a surface buoyancy in a rapidly rotating, stratified potential vorticity fluid. In the first part of the talk, we will present some global regularity results for its dissipative analogue in the critical regime for the two dimensional sphere.

After that, we will introduce the Incompressible Slice Model dealing with oceanic and atmospheric fluid motions taking place in a vertical slice domain $\Omega \subset \mathbb{R}^2$, with smooth boundary. The ISM can be understood as a toy model for the full 3D Euler-Boussinesq equations. We will study the solution properties of the Incompressible Slice Model: characterizing a class of equilibrium solutions, establishing the local existence of solutions and providing a blow-up criterion.

Introduction: abstract and conclusions

This thesis centers on the study of two different problems of partial differential equations arising from geophysics and fluid mechanics: the surface quasi-geostrophic equation and the so called, Incompressible Slice Model. Geophysical fluid dynamics, in its broadest sense, is the study of fluid motions in the earth and other planets. In particular, the behaviour of the oceans and atmosphere, [Fri80; Ped87; Bat99]. Besides the relevance to geophysics, the subject is enriching and appealing from a mathematical point of view because the PDE's which arise frequently display interesting properties.

The starting point of geophysical fluid dynamics assumes that the dynamics of the motions are determined by the equations of continuum, which in an inertial or non rotating frame are given by the condition of mass conservation, i.e. the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho u = 0, \quad (1.1)$$

where ρ is the density and u the vector velocity, and Newton's law of motion

$$\rho \frac{Du}{Dt} = -\nabla p + \rho \nabla \Phi + \mathcal{F}(u), \quad (1.2)$$

where p is the pressure, Φ the potential and \mathcal{F} represents the frictional force of the fluid. Here $\frac{D}{Dt} = \partial_t + u \cdot \nabla$ represents the material derivative. To close the dynamics of the system, we also need to consider the first law of thermodynamics which reads

$$\rho \frac{De}{Dt} = -p \rho \frac{D}{Dt} \rho^{-1} + \kappa \Delta T + \rho Q, \quad (1.3)$$

where e is the internal energy, T is the temperature, κ is the thermal conductivity and Q is the rate of heat addition per unit mass.

Equations (1.1)-(1.3) describe the dynamics of a fluid on a non rotating coordinate frame of reference. However, to be more precise, we need to take into account the angular rotation Ω coming from planetary rotation. Therefore, the most natural way to do this, is to change and interpret the equations from a new rotating coordinate frame. Naturally, the phenomena themselves are unaltered by the choice of the frame reference, but the description of the phenomena depends on our choice. For instance, Newton's law of motion becomes

$$\rho \left[\frac{Du}{Dt} + 2\Omega \times u \right] = -\nabla p + \rho \nabla \Phi + \mathcal{F}(u), \quad (1.4)$$

where the term $2\Omega \times u$ is the Coriolis force or acceleration.

Taking into account these fundamental laws and the different ways to describe them, let us continue reviewing in more detail the models that we will focus on.

The surface quasi-geostrophic equation

Among the wide range of problems arising in geophysics, we will be concerned with those for which the length scale is sufficiently large that the planetary rotation has a significant effect on the dynamics of the fluid, called large scale phenomena. This can be quantified directly by inspecting the importance of the Coriolis force in (1.4). Its order of magnitude can be estimated as

$$2\Omega \times u = \mathcal{O}(2\Omega[U]),$$

where $[U]$ is the velocity length scale, while the relative acceleration

$$\frac{Du}{Dt} = \mathcal{O}\left(\frac{[U]^2}{[L]}\right), \quad \text{assuming } \partial_t u \sim u \cdot \nabla u,$$

with $[L]$ the characteristic length scale. Therefore, the nature of the perceived acceleration on a rotating frame depends on the relative size of both quantities,

$$\epsilon = \frac{[U]}{2\Omega[L]},$$

known as the Rossby number. Large scale flows are defined as those with sufficiently large $[L]$ such that ϵ is smaller or equal than one (in the specific case of the Earth, $\epsilon = 0.137$.) The first model we will deal with illustrates the behaviour of long scale dynamics of the atmosphere in mid-latitudes ($\epsilon \ll 1$) which are usually governed by the geostrophic approximation, [Ped87]. This approximation is described by the balance between the Coriolis force and the pressure gradients

$$\begin{cases} -fu_2 = \frac{\partial p}{\partial x} \\ -fu_1 = \frac{\partial p}{\partial y} \end{cases}$$

where $u = (u_1, u_2)$ is the velocity field, p is the pressure and f is the Coriolis force (assumed constant). Therefore, by the above equation, we can identify the pressure field as the stream function. Notice that these equations do not provide any information on the dynamics of the pressure, so one needs to go further in the

expansion.¹ In the next order, one can find non trivial dynamics described by the quasi-geostrophic equation [Cha71; Blu82; Ped87], which reads

$$\partial_t q + u \cdot \nabla q = 0. \quad (1.5)$$

The potential vorticity q can be rewritten in terms of the stream function

$$q = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial}{\partial z} \left(\left(\frac{f}{N} \right)^2 \frac{\partial \psi}{\partial z} \right),$$

where N is the buoyancy frequency or Brunt-Väisälä number. To derive the desired model, we need to do further assumptions and simplifications. First, let us assume that N is constant and after rescaling the z variable,

$$q = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \Delta_{x,y,z} \psi.$$

Next, assume that the initial vorticity q_0 is zero, and by (1.5) zero for all times, therefore obtaining

$$\Delta_{x,y,z} \psi = 0. \quad (1.6)$$

Focusing on the Earth surface, assumed flat, the velocity field is purely horizontal and identifying the pressure field as the stream function

$$u = (u_1, u_2) = \nabla^\perp \psi, \quad (1.7)$$

where $\nabla^\perp = (-\partial_y, \partial_x)$. Moreover applying the Boussinesq approximation, we infer that the vertical derivative of the pressure (or stream function) is proportional to the temperature

$$\theta \propto \frac{\partial \psi}{\partial z}. \quad (1.8)$$

From the energy balance law, we have that

$$\partial_t \theta + u \cdot \nabla \theta = 0. \quad (1.9)$$

Since ψ satisfies the Laplace equation (1.6) the vertical derivative in (1.8) is related via the Dirichlet-to-Neumann map to

$$\theta = \frac{\partial \psi}{\partial z} = (-\Delta_{x,y})^{\frac{1}{2}} \psi, \quad (1.10)$$

¹Formally, this is done by using an asymptotic expansion and using the smallness of the Rossby number ϵ , [Ped87].

where $(-\Delta_{x,y})^{\frac{1}{2}}$ is the 2D fractional laplacian. In conclusion, from (1.7), (1.9) and (1.10) we finally obtain the following system of equations

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp \psi, \\ (-\Delta)^{\frac{1}{2}} \psi = \theta, \end{cases}$$

known as the **surface quasi-geostrophic equation (SQG)**.

As we have seen, this system is derived from the quasi-geostrophic equation when we make certain simplifications in the geometry and approximate certain quantities. It is clear that the main objective of this model is not to obtain an exact description of the phenomenon but to provide the essence of it. The surface quasi-geostrophic equation is a two dimensional nonlocal partial differential equation of geophysical importance, describing the evolution of a surface buoyancy in a rapidly rotating, stratified potential vorticity fluid [Ped87; Con+94; Hel+95]. We can rewrite the equation more compactly as

$$(SQG) : \begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = R^\perp \theta, \end{cases}$$

since the vector field $u(x, t)$ is given by

$$u = \nabla^\perp \psi = \nabla^\perp (-\Delta)^{-\frac{1}{2}} \theta = R^\perp \theta = (-R_2 \theta, R_1 \theta),$$

where R_j are the Riesz transforms for $j = 1, 2$ and the potential temperature is a scalar $\theta(x, t)$ with $x \in \mathbb{R}^2$, $t \geq 0$. Beyond its own physical interest, the SQG equation serves as a toy model for the well-known incompressible 3D Euler equation, [Con+94]. For an incompressible fluid, the 3D Euler equation, reads

$$(E) : \begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p, \\ \operatorname{div} u = 0, \end{cases}$$

where $u = (u_1, u_2, u_3)$ is the velocity field of the fluid and $p = (p_1, p_2, p_3)$ the pressure. Taking the rotational $(\nabla \times)$ to the equation, we have that

$$(E_\omega) : \begin{cases} \partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u, \\ \omega = \nabla \times u, \end{cases}$$

known as the Euler equation in vorticity form. On the other hand, taking the perpendicular gradient on the (SQG) , we have that

$$(SQG_{\nabla^\perp}) : \begin{cases} \partial_t \nabla^\perp \theta + u \cdot \nabla \nabla^\perp \theta = \nabla^\perp \theta \cdot \nabla u, \\ \operatorname{div} u = 0. \end{cases}$$

Now is easy to check that the vorticity ω in the Euler equation and the perpendicular gradient of the potential temperature $\nabla^\perp \theta$ play essentially the same role. In both cases, the transport is affected by a non-linear quadratic interaction: $\omega \cdot \nabla u$ in the case of (E_ω) and $\nabla^\perp \theta \cdot \nabla u$ for (SQG_{∇^\perp}) . The gradient of the velocity ∇u can be related with ω and $\nabla^\perp \theta$, through a singular operator with zero order Fourier symbol. The analytic and geometric analogies (cf. [Con+94]) between both equations, suggest that a good understanding of the solutions of the SQG, may shed some light to understand the 3D Euler equation.

Initial numerical simulations presented evidence of fast growth of the gradient of the SQG when the geometry of the level sets contain a hyperbolic saddle, [Con+94], suggesting the possible formation of sharp fronts [OY97; Con+98]. Later, several analytical studies on the suggested singular scenerios were carried out in [C98; CF01; CF02; FR11]. Despite the great effort of the mathematical community, whether solutions of the SQG can develop singularities in finite-time is an intriguing open problem.

When fractional dissipation is added to the equation, we have

$$(SQG_\alpha) : \begin{cases} \partial_t \theta + u \cdot \nabla \theta + \nu (-\Delta)^\alpha \theta = 0, \\ u = R^\perp \theta, \end{cases}$$

known as the dissipative surface quasi geostrophic equation. Here $\nu > 0$ is the viscosity coefficient and $\alpha \in [0, 1]$. The non local operator $(-\Delta)^\alpha$ is defined using the Fourier transform as

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi),$$

where \widehat{f} denotes the Fourier transform of f . According to the value of the parameter α we can distinguish three regimes: the supercritical case $0 \leq \alpha < \frac{1}{2}$, the critical case $\alpha = \frac{1}{2}$ and the subcritical case $\frac{1}{2} < \alpha \leq 1$. Criticality can be understood as a struggle between the non-linear and dissipative term.² Indeed, taking Fourier transform in space in (SQG_α) , we have that

$$\partial_t \hat{\theta}(l, t) + \nu |l|^{2\alpha} \hat{\theta}(l, t) = - \sum_{j+k=l} \frac{j^\perp \cdot k}{|j|} \hat{\theta}(j, t) \hat{\theta}(k, l). \quad (1.11)$$

²One can also understand the criticality of the equation by the scale invariance and available conservation laws. However, here the criticality is in the sense of *Goldilocks*, [Con17].

Now it is easy to see that the non-linear term and the dissipative term are comparable when $\alpha = \frac{1}{2}$. However, when $\alpha > \frac{1}{2}$ the dissipative term is stronger than the non linear term and the other way when $\alpha < \frac{1}{2}$. The problem about the global existence and the study of finite-time singularities for the (SQG_α) , has been investigated and analyzed extensively in the last twenty years.

The subcritical case ($\alpha < \frac{1}{2}$) is understood and resolved in [Res95; CW99], where the global existence of smooth solutions is proved via perturbative arguments, interpolation and energy estimates. Qualitative properties of weak and strong solutions for the subcritical case are also studied in [Wu01; Ju04; DL08].

The critical case ($\alpha = 1$) is much more challenging due to the balance between the non-linear term and the dissipation. Perturbative methods are not useful anymore and more refined strategies are needed to tackle the problem. Let us take a closer look at the techniques that were developed, since part of this memoir focuses on understanding the critical SQG equation in other physical interesting settings. In [Con+01], the authors proved the global existence of solutions for small initial data. In particular, they showed that if $\|\theta_0\|_{L^\infty}$ is small enough, then for $\theta_0 \in H^1$ the solution remains in H^1 globally. See also [CL03; CC04; Miu06; HK07] for global regularity results with smallness hypothesis in other critical spaces.

Independently and with different approaches [Kis+07] and [CV10a], showed the global existence of solutions for arbitrarily large initial data. The proofs are different in spirit and flavour. The one in [Kis+07] is based on building a certain family of modulus of continuity, so that if the data obeys this modulus, the evolution preserves it for all time. This robust technique has been used to solve different variants of the problem: in the presence of an external force [Fri+09], a linear dispersive term [KN10], diffusion reaction equation [SV12] and singular generalizations of the equation [LX18].

On the other hand, the strategy in [CV10a] is based on the ideas used by E. De Giorgi to solve the nineteenth problem of Hilbert [DG57], adapted to the non local and parabolic character of the equation. More precisely, they prove that any weak solution becomes instantaneously Hölder continuous and therefore regular for all time. The generality of the proof allows them to treat general transport diffusion equations in any dimension where the drift $u \in BMO_x$. See also [CV10b; Caf+11; Sil12; Sil+13] for applications of De Giorgi's scheme to other transport-diffusion equations.

In [KN09], the authors gave a third proof different from the previous ones, using special test functions in order to control the evolution of the Hölder norm.

Finally, a fourth and final strategy was introduced by [CV12], using non-linear lower bounds for non-local operators such as the fractional Laplacian. These inequalities demonstrate quantitatively the dominance of the dissipative term over the non-

linear term, which is not apparently visible in the other proofs. To show the global regularity of the critical SQG, they combined the lower bounds with an additional hypothesis called *only small shocks* (OSS) condition. A variant of this argument in [Con+15], allows to replace the condition of (OSS) using the method of De Giorgi.

For the moment, the problem of global existence of smooth solutions or finite-time singularities for the supercritical case ($\alpha > \frac{1}{2}$), is completely open. Only global existence is known in the case of small initial data [CL03; CC04; Wu04; Miu06; Yu08] and the eventual regularity of solutions, [Sil10; Dab11].

The critical surface quasi-geostrophic equation on the two dimensional sphere

In the derivation of the SQG we have used several approximations and simplifications. For instance, we have assumed the Earth's flatness neglecting the curvature. The second chapter of this thesis, relies on understanding the critical surface quasi-geostrophic equation on more physical relevant situations (i.e. Earth's surface) where curvature is present in a natural way. Therefore, let us consider the equation in a compact orientable surface M with riemannian metric g given by

$$(SQG_M) : \begin{cases} \partial_t \theta + u \cdot \nabla_g \theta + \Lambda_g \theta = 0, \\ u = \nabla_g^\perp \Lambda_g^{-1} \theta = R_g^\perp \theta, \end{cases} \quad (1.12)$$

where Λ_g is the square root of the Laplace-Beltrami operator $-\Delta_g$. The main result of this chapter, is devoted to study the equation in the two dimensional sphere:

Theorem 1.1 ([AO+18a]). *Let $\theta_0 \in L^2(\mathbb{S}^2)$ and θ a weak solution of the following Cauchy problem*

$$(SQG_{\mathbb{S}^2}) : \begin{cases} \partial_t \theta + u \cdot \nabla_g \theta + \Lambda_g \theta = 0, \\ \theta(x, 0) = \theta_0, \end{cases} \quad (1.13)$$

where $u = \nabla_g^\perp \Lambda_g^{-1} \theta$. Then $\theta(x, t)$ is continuous with an explicit modulus of continuity for every $t > 0$.

In the particular case of the two dimensional sphere an explicit computation shows $\operatorname{div}_g u = 0$. However, this fact also holds on any two dimensional Riemannian manifold. The higher dimensional analogue is more delicate, but in even dimensions and in the presence of a symplectic structure, there is a canonical construction of an orthogonal gradient such that its divergence vanishes.

The idea of the proof follows to some extent the strategy introduced in [CV10a], using a non local version of De Giorgi's method. As we mentioned before, their result can be applied to drift-diffusion equations whose divergence velocity field has bounded mean oscillation (BMO). However, our situation is not that fortunate

because curvature matters. In a nutshell lack of scaling makes the analysis much harder in our case.

Grosso modo De Giorgi's iteration, or the variant we used to tackle this problem, is based on improving the regularity of the solution from L^2 to C^α . The later functional space is equipped with the norm

$$\|f\|_{C^\alpha} := \|f\|_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_g(x, y)^\alpha}.$$

Following this strategy, the first step is to prove the global existence of weak solutions to the (SQG_M) for initial data $\theta_0 \in L_x^2$, cf. Appendix A. The next step, consists in improving the regularity from L_x^2 to L_x^∞ . To that purpose, in Section 2.1 we prove uniform bounds for the essential supremum of a global weak solution in space and strictly positive time, i.e.,

$$|\theta(x, t)| \leq C(M, t_0, \|\theta_0\|_{L^2(M)}), \quad (1.14)$$

for every $x \in M$ and $t \geq t_0 > 0$. We achieve this using a nonlinear recurrence for consecutive energy truncations based on the interplay between a global energy inequality and the Sobolev inequality. Therefore by (1.14), we know that $\theta \in L_t^\infty L_x^\infty$, and hence, $u = R_g^\perp \theta \in L_t^\infty L_x^p$ for all $p \in [2, \infty)$. Once this is proven, we may treat the problem as if it was linear, forgetting about the θ dependence of u .

In Section 2.2, we provide two technical lemmas which prove a quantitative maximum principle for certain family of barriers adapted to the local geometry. The constructed barrier are crucial in order to prove the C^α regularity. Actually, to gain the desired regularity we will have to control the oscillation decay³.

Section 2.3 is devoted to present a local energy inequality which combined with the constructed barriers, will provide a non linear recurrence using De Giorgi's scheme yielding the oscillation decay. However, to show this local energy inequality (cf. Lemma 2.3) we need to impose a stronger condition to the drift u that is not satisfied by the (SQG_M) . Therefore, at this point, the rest of the argument (cf. Section 2.4) will be devoted to prove an oscillation decay, yielding the next result:

Theorem 1.2 ([AO+18a]). *Let $\theta_0 \in L^2(M)$ and θ a weak solution of the following Cauchy problem*

$$\begin{cases} \partial_t \theta + u \cdot \nabla_g \theta + \Lambda_g \theta = 0, \\ \theta(x, 0) = \theta_0, \end{cases}$$

where the divergence free velocity field $u \in L^\infty(M)$ uniformly in time. Then $\theta(x, t)$ is of class C^α for any $t > 0$.

³It is well known that that a proper control in the oscillation decay, implies directly the C^α regularity. We will not include this digression here, in order to simplify the exposition

One may think of Theorem 1.2 as a subcritical version of the problem, since the hypothesis is satisfied when $u = \nabla_g^\perp \Lambda_g^{-1-\epsilon} \theta$ for $\epsilon > 0$. In the euclidean setting this result has been proved in [Sil12] without the divergence free condition. His proof relies on some quantitative comparison principles which are hard to adapt in our context.

Section 2.4, is divided into three parts. First in Subsection 2.4.1, an oscillation decay of θ will be achieved provided the initial energy is small enough (cf. Proposition 2.2). In Subsection 2.4.2, we will drop the small mean energy condition using a non local version of De Giorgi's isoperimetric inequality (cf. Lemma 2.6). Taking all the above into account, we prove Theorem 1.2 in Subsection 2.4.3.

Finally in Section 2.5, we prove Theorem 1.1 by modifying the arguments used previously to show Theorem 1.2, where the velocity field does not satisfy a priori the hypothesis of the local energy inequality (cf. (2.3)). To that purpose, we adapt with a geometrical argument the strategy of [CV10a], relying on rescalings and translations. In our case, using the group of rotations of the sphere, we are able to show a logarithmic modulus of continuity.

Global existence of strong solutions to the critical surface quasi-geostrophic equation

In Chapter 3, we will continue studying the critical surface quasi-geostrophic equation (1.12), but in this case we will be interested in the problem of global well-posedness in Sobolev spaces. More precisely, we will prove the following result:

Theorem 1.3 ([AO+18b]). *Given an initial data $\theta_0 \in H^s(\mathbb{S}^2)$ with $s > \frac{3}{2}$, there exists a global solution $\theta(x, t)$ of $(SQG_{\mathbb{S}^2})$ in $H^s(\mathbb{S}^2)$. Moreover, the solution becomes instantaneously regular.*

The general structure is based on combining three basic ingredients: first an integral representation of the fractional Laplace-Beltrami operator on a general compact manifold.⁴ Next, we will follow and adapt the strategy in [CV12] which is based on a nonlinear maximum principle established with help of the aforementioned explicit integral representation for the fractional Laplace- Beltrami operator. The last ingredient, is to combine the modulus of continuity given by Theorem 1.1 which implies what in [CV12] define as *only small shocks condition*.

The strategy of the proof follows the following lines:

In Section 3.1 we provide several observations that will be instrumental in the sequel. In particular, we introduce the integral representation and set a particular system of local coordinates we will take advantage of based on rotations and the stereographic

⁴This result of independent interest, joint work with A. Córdoba and A.D. Martínez in [AO+18c], is part of the PhD thesis of last author.

projection (cf. Figure (3.1)). The appropriate coordinates, will be very useful to prove certain estimates and commutators of non local operators (cf. Lemma 3.1).

Once the tools are sharpened, in Section 3.2 we prove a non linear lower bound for the fractional Laplace-Beltrami operator as in [CV12]. This can be understood as a refinement of the well-known Córdoba-Córdoba pointwise inequality [CC04; CM15]. However, we need to deal with some additional difficulties due to the curvature effects.

Finally, in Section 3.3, we will provide a global in time bound for the gradient of the solution in L^∞ (cf. Proposition 3.2). To that purpose we perform pointwise estimates combined with the beforementioned lower bounds and Theorem 1.1. This global in time control implies immediately Theorem 1.3 (cf. Appendix B).

Slice Models

Inside the broad area of atmospheric science, apart from the surface quasi-geostrophic equation, there are a wide variety of models to investigate. For instance, the so called Slice Models are frequently used to study the behaviour of weather and specifically the formation of atmospheric fronts, fundamental in meteorology [Cul07; Vis14].

The Cotter-Holm Slice Model (CHSM) was introduced in [CH13] for oceanic and atmospheric fluid motions taking place in a vertical slice domain $\Omega \subset \mathbb{R}^2$, with smooth boundary $\partial\Omega$. The fluid motion in the vertical slice is coupled dynamically to the flow velocity transverse to the slice, which is assumed to vary linearly with distance normal to the slice. This assumption about the transverse flow through the vertical slice in the CHSM simplifies its mathematics while still capturing an important aspect of the 3D flow. The slice framework has been designed for the study of weather fronts; see [Yam+17].

In [HB71], fronts were described mathematically and a general theory for studying fronts was developed. The necessary assumptions made are similar to the ones in CHSM. In general, slice models are used to study front formation with geostrophic balance in the cross-front direction. This assumption simplifies the analysis by formulating the dynamics in a two-dimensional slice, while still providing some realistic results.

Front formation is directly related to baroclinic instability. Eady considered a classical model in 1949 (c.f. [Ead49]) in order to study the effects of baroclinic instability. Decades of observation have concluded that the most important source of synoptic scale variations in the atmosphere is due to the so called baroclinic instability. In [Bad+09], this is linked to frontal systems and it is shown to trigger the formation of eddies in the North Sea.

The Incompressible Slice Model

In Chapter 4 we study a particular case of the (CHSM) known as the Incompressible Slice Model (ISM) ⁵. It should be emphasized that the ISM is potentially useful in numerical simulations of fronts. For instance, since the domain consists of a two-dimensional slice, computer simulations take much less time to run than a full three-dimensional model. There have been many studies on this kind of idealized models to predict the formation and evolution of weather fronts (c.f. [NH89; Bud+13; Vis14; Vis+14; Yam+17]).

The ISM evolution equations for fluid velocity components $u_S(x, z, t) : \Omega \subset \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$, scalar $u_T(x, z, t) : \Omega \subset \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}$ transverse to the slice, as well as the potential temperature $\theta_S(x, z, t) : \Omega \subset \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}$, are given by

$$\partial_t u_S + u_S \cdot \nabla u_S - f u_T \hat{x} = -\nabla p + \frac{g}{\theta_0} \theta_S \hat{z}, \quad (1.15)$$

$$\partial_t u_T + u_S \cdot \nabla u_T + f u_S \cdot \hat{x} = -\frac{g}{\theta_0} z s, \quad (1.16)$$

$$\partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T s = 0, \quad (1.17)$$

$$\nabla \cdot u_S = 0. \quad (1.18)$$

Here g is the acceleration due to gravity, θ_0 is the reference temperature, f is the Coriolis parameter, which is assumed to be a constant, and s is a constant which measures the variation of the potential temperature in the transverse direction. In these equations, ∇ denotes the 2D gradient in the slice, p is the pressure obtained from incompressibility of the flow in the slice ($\nabla \cdot u_S = 0$), while \hat{x} and \hat{z} denote horizontal and vertical unit vectors in the slice. The flow is taken to be tangent to the boundary, so that

$$u_S \cdot n = 0 \text{ on } \partial\Omega, \quad (1.19)$$

and n is the outward unit normal vector to the boundary $\partial\Omega$.

The ISM resembles the standard 2D Boussinesq equations, which are commonly used to model large scale atmospheric and oceanic flows that are responsible for cold fronts and the jet stream [Ped87]. The Boussinesq equations have been widely studied and considerable attention has been dedicated recently to their well-posedness and regularity [CD80; HL05; Cha06]. However, the fundamental question of whether their classical solutions blow up in finite time remains open. This problem is even discussed in Yudovich's *eleven great problems of mathematical hydrodynamics* [Yud03]. Important progress in the global regularity problem has been made by Luo and Hou [LH14b; LH14a], who have produced strong numerical evidence that smooth solutions of the 3D axisymmetric Euler equation system, which can be identified with the inviscid 2D Boussinesq equation, develop a singularity in finite time when the fluid domain has a solid boundary. Recently, the authors in [EJ18], have shown

⁵This model is also known in the literature as Euler-Boussinesq Eady model

finite-time singularity formation for strong solutions of the 2D Boussinesq system when the fluid domain is a sector of angle less than π .

The goal of this chapter is threefold:

First, we provide a characterisation of a restricted class of equilibrium solutions of the ISM, and study the model's stability of solutions around it. The results can be stated as follows:

Theorem 1.4 ([AOL18b]). *A restricted class of stationary solutions of the ISM equations (1.15)-(1.18) with boundary condition (1.19) is given by critical points of the generalized Hamiltonian*

$$H_\Phi = \int_\Omega \left\{ \frac{1}{2}(|u_S|^2 + u_T^2) - \gamma_S \theta_S \right\} dV + \int_\Omega \Phi(q) dV + \sum_{i=0}^n a_i \int_{\partial\Omega_i} v_S \cdot ds.$$

These are given by the conditions

$$\begin{aligned} a_i &= \Phi'(q_e|_{\partial\Omega_i}), \quad \text{for } i = 0, \dots, n, \\ u_{Se} &= -\text{curl}(\Phi'(q_e)\hat{y})s, \\ u_{Te} &= \text{curl}(\Phi'(q_e)\hat{y}) \cdot \nabla\theta_{Se}, \\ \gamma_S &= \text{curl}(\Phi'(q_e)\hat{y}) \cdot (\nabla u_{Te} + f\hat{x}). \end{aligned}$$

Here $\gamma_S = (g/\theta_0)z$, $v_S = su_S - (u_T + fx)\nabla\theta_S$ is the circulation velocity in the ISM, and $q = \text{curl}(v_S) \cdot \hat{y}$ is the potential vorticity. Moreover, Φ can be written in terms of the Bernoulli function K for the stationary solution as

$$\Phi(\lambda) = \lambda \left(\int_\lambda \frac{K(t)}{t^2} dt + C \right).$$

Theorem 1.5 ([AOL18b]). *An equilibrium point of the ISM belonging to the restricted class specified in Theorem 1.4 is formally stable if*

$$\frac{(\hat{z} \times \nabla q_e) \cdot u_{Se}}{|\nabla q_e|^2} > 0. \quad (1.20)$$

This last result mimics the first Arnold's Theorem of formal stability for two-dimensional incompressible Euler [Arn89], where the formal stability condition reads

$$\frac{(\hat{z} \times \nabla \omega_e) \cdot u_e}{|\nabla \omega_e|^2} > 0.$$

Here $\omega_e = \text{curl}(u_e) \cdot \hat{z}$. The extra term $q_R = -\text{curl}((u_T + fx)\nabla\theta_S)$ appearing in (1.20) is due to the introduction of a transverse velocity u_T and a potential temperature θ_S .

Theorem 1.6 ([AOL18b]). *We can define a norm Q on $\mathfrak{X}(\Omega) \otimes \mathcal{F}(\Omega) \times \wedge^2(\Omega)$ such that an equilibrium point of the ISM belonging to the restricted class specified in Theorem 1.4 is nonlinearly stable with respect to Q if*

$$0 < \lambda_1 \leq \frac{(\hat{z} \times \nabla q_e) \cdot u_{Se}}{|\nabla q_e|^2} \leq \lambda_2 < \infty.$$

Next, we establish the local well-posedness of the ISM in Sobolev spaces:

Theorem 1.7 ([AOL18b]). *For $s > 2$ integer and initial data $(u_S^0, u_T^0, \theta_S^0) \in H_*^s(\Omega) \times H^s(\Omega) \times H^s(\Omega)$, there exists a time $T = T(\|(u_S^0, u_T^0, \theta_S^0)\|_{H^s}) > 0$ such that the ISM equations (1.15)-(1.18) with boundary condition (1.19) have a unique solution (u_S, u_T, θ_S) in $C([0, T]; H_*^s(\Omega) \times H^s(\Omega) \times H^s(\Omega))$.*

Lastly, we also prove a blow-up criterion for the Incompressible Slice Model, which reads as follows:

Theorem 1.8 ([AOL18b]). *Suppose that $(u_S^0, u_T^0, \theta_S^0) \in H_*^s(\Omega) \times H^s(\Omega) \times H^s(\Omega)$ for $s > 2$ an integer and that the solution (u_S, u_T, θ_S) of equations (1.15)-(1.18) with boundary condition (1.19) is of class $C([0, T]; H_*^s(\Omega) \times H^s(\Omega) \times H^s(\Omega))$. Then for $T^* < \infty$, the following two statements are equivalent:*

$$(i) \quad E(t) < \infty, \quad \forall t < T^* \quad \text{and} \quad \limsup_{t \rightarrow T^*} E(t) = \infty, \quad (1.21)$$

$$(ii) \quad \int_0^t \|\nabla u_S(s)\|_{L^\infty} ds < \infty, \forall t < T^* \text{ and } \int_0^{T^*} \|\nabla u_S(s)\|_{L^\infty} ds = \infty \quad (1.22)$$

where $E(t) = \|u_S\|_{H^s}^2 + \|u_T\|_{H^s}^2 + \|\theta_S\|_{H^s}^2$. If such T^* exists then T^* is called the first-time blow-up and (1.22) is a blow-up criterion.

The chapter is structured along the following lines:

In Section 4.1 we introduce some basic definitions and well-known lemmas about Sobolev spaces. We also include several preliminary results regarding Arnold's stability Theorem and the Kato-Lai Theorem for nonlinear evolution equations.

Section 4.2 introduces the Cotter-Holm Slice Model by using its Lagrangian formulation, as carried out in [CH13]. In particular, in Subsection 4.2.3 we substitute the Euler-Boussinesq Lagrangian and derive the ISM equations, where we focus our interest. Since the ISM equations are Euler-Poincaré equations, they enjoy fundamental conservation laws (Kelvin's circulation, total energy, potential vorticity...) which we mention in Subsection 4.2.4.

In Section 4.3, we characterize a class of equilibrium solutions of the ISM, and study formal and nonlinear stability around them by applying the Energy-Casimir method [Hol+85]. To that purpose, we first characterize a class of equilibrium solutions of

the ISM in Subsection 4.11. After that, we study the formal and nonlinear stability around them in Subsections 4.3.2 and 4.3.3, respectively.

In Section 4.4 we show the local well-posedness of the Incompressible Slice Model, by adapting an abstract result for systems of nonlinear equations [KL84]. Before starting with the proof, we demonstrate some estimates and make some observations. We prove existence, uniqueness, and regularity of solutions, in Subsection 4.4.1 concluding Theorem 1.7.

Finally in Section 4.5 we construct a continuation type criterion which is well-known, for instance, for the Euler equation (see [Bea+84]), proving Theorem 1.8.

Concluding remarks and observations

This thesis is devoted to the study of non linear partial differential equations which describe some phenomena arising in geophysical fluid dynamics. In particular, we have focused our attention in two different models. First, we have proved the global existence of solutions of the critical surface quasi-geostrophic equation on the sphere. Then, we have study the solution properties of the Incompressible Slice Model: characterizing a class of equilibrium solutions, establishing the local existence of solutions and providing a blow-up criterion.

The main results of this thesis have been published in the following scientific journals:

- D. Alonso-Orán, A. Córdoba and A. D. Martínez, *Continuity of weak solutions of the critical surface quasigeostrophic equation on \mathbb{S}^2* , Advances in Mathematics, **328** (2018), pp. 264–299. ISSN 0001-8708, <https://doi.org/10.1016/j.aim.2018.01.015>. (Chapter 2)
- D. Alonso-Orán, A. Córdoba and A. D. Martínez, *Global well-posedness of critical surface quasigeostrophic equation on the sphere*, Advances in Mathematics, **328** (2018), pp. 248–263. ISSN 0001-8708, <https://doi.org/10.1016/j.aim.2018.01.016>. (Chapter 3)
- D. Alonso-Orán and A. Bethancourt de León, *Stability, well-posedness and blow-up criterion for the Incompressible Slice Model*, Physica D: Nonlinear Phenomena, 2018, ISSN 0167-2789, <https://doi.org/10.1016/j.physd.2018.12.005>. (Chapter 4)

Moreover in the Appendix C, we have described briefly some other results that the candidate has obtained during the PhD. We did not include them in the central part of this thesis to homogenize as far as possible the main line of research. the results are collected in the following articles:

- D. Alonso-Orán and A. Bethancourt de León and S. Takao, *The Burgers' equation with stochastic transport: shock formation, local and global existence of smooth solutions*, arXiv:1808.07821, 2018.
- D. Alonso-Orán, F. Chamizo, A. D. Martínez and A. Mas, *Pointwise monotonicity of heat kernels*, arXiv:1807.11072, 2018.
- D. Alonso-Orán and A. Bethancourt de León, *On the well-posedness of stochastic Boussinesq equations with cylindrical multiplicative noise*, arXiv:1807.09493, 2018.

Introducción: resumen y conclusiones

Esta memoria se centra en el estudio de dos problemas de ecuaciones en derivadas parciales que provienen de la geofísica y la mecánica de fluidos: la ecuación cuasi-geostrófica superficial (SQG) y el *Incompressible Slice Model* (ISM). La dinámica de fluidos geofísicos estudia el movimiento de los fluidos en la tierra y otros planetas, en particular el comportamiento de la atmósfera y los océanos, [Fri80; Ped87; Bat99]. Partiendo de la premisa de que la geofísica de fluidos se rige por las ecuaciones de movimiento de los medios continuos, debemos tener en cuenta sus leyes fundamentales. Estas son:

La ecuación de la continuidad

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho u = 0, \quad (1.23)$$

donde ρ es la densidad y u el vector de velocidad; la ley del movimiento de Newton

$$\rho \frac{Du}{Dt} = -\nabla p + \rho \nabla \Phi + \mathcal{F}(u), \quad (1.24)$$

donde p es la presión, Φ el potencial y \mathcal{F} representa de fricción del fluido; y la primera ley de la termodinámica

$$\rho \frac{De}{Dt} = -p \rho \frac{D}{Dt} \rho^{-1} + k \nabla^2 T + \rho Q. \quad (1.25)$$

Aquí e es la energía interna, T la temperatura, κ la conductividad térmica y Q el ratio de calor por unidad de masa debido a fuentes externas de calor. Las ecuaciones (1.23)-(1.25) describen el movimiento de los fluidos en un sistema de referencia fijo. Sin embargo, si queremos ser más precisos, debemos tener en cuenta la rotación angular Ω de nuestro planeta. Esta rotación genera una fuerza de aceleración que afectará a la dinámica de los fluidos que estudiemos, conocida como aceleración o fuerza de Coriolis. Por tanto, cuando observemos estos fenómenos desde este nuevo

sistema de referencia rotatorio, las ecuaciones se verán afectadas. Por ejemplo, la ecuación de Newton (1.24) vendrá descrita por

$$\rho \frac{Du}{Dt} + 2\Omega \times u = -\nabla p + \rho \nabla \Phi + \mathcal{F}(u), \quad (1.26)$$

donde el término $2\Omega \times u$ es la fuerza de Coriolis.

Teniendo en cuenta estas leyes elementales y las distintas maneras de describirlas, veamos a continuación con más detalle cuáles son las propiedades y características de los modelos que estudiaremos a lo largo de esta tesis.

La ecuación cuasi-geostrófica superficial

Dentro del amplio abanico que comprende la dinámica de fluidos geofísicos, estamos interesados en estudiar los fenómenos de gran escala, donde podemos situar el primero de nuestros problemas. Diremos que un fenómeno es de gran escala si su dinámica se ve afectada de manera notoria por la rotación planetaria. Matemáticamente cuantificamos esto a través del número de Rossby, que caracteriza el cociente entre la aceleración de un fluido y la fuerza de Coriolis,

$$\epsilon = \frac{[U]}{2\Omega[L]},$$

donde $[U]$ es la escala velocidad horizontal y $[L]$ es la escala de la longitud horizontal. Por tanto, para números de Rossby pequeños, $\epsilon < 1$, estamos ante un fenómeno de gran escala (en el caso particular de la Tierra, $\epsilon = 0.137$). El modelo que nos interesa estudiar, tiene como principal característica una rápida rotación ($\epsilon \ll 1$) dando lugar a la aproximación geostrófica. Esta consiste en el equilibrio entre las fuerzas de Coriolis y el gradiente de presión, dado por

$$\begin{cases} -fu_2 = \frac{\partial p}{\partial x}, \\ -fu_1 = \frac{\partial p}{\partial y}, \end{cases}$$

donde $u = (u_1, u_2)$ es el campo de velocidades, p es la presión y f es la fuerza de Coriolis (que asumimos constante). Como podemos observar, esta primera aproximación no da información sobre la dinámica del fluido y por tanto es necesario ir al siguiente orden.⁶ En este orden, ya podemos encontrar dinámicas no triviales

⁶ Formalmente, esto se hace utilizando una expansión asintótica mediante métodos de análisis de escala explotando la pequeñez del número de Rossby.

descritas por las ecuaciones cuasi-geostróficas, dadas por la conservación de la vorticidad potencial q [Cha71; Blu82; Ped87],

$$\partial_t q + u \cdot \nabla q = 0. \quad (1.27)$$

Este potencial se puede describir a través de una función corriente y en el caso general esta dado por

$$q = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial}{\partial z} \left(\left(\frac{f}{N} \right)^2 \frac{\partial \psi}{\partial z} \right),$$

donde N es la llamada frecuencia de Brunt-Väisälä o frecuencia de flotabilidad. Sin embargo, para llegar a derivar el modelo que trata parte de esta memoria, necesitamos alguna simplificación más. En primer lugar, supongamos que N es constante y por tanto, tras reescalar la variable z ,

$$q = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \Delta_{x,y,z} \psi.$$

Luego, supongamos que la vorticidad potencial inicial q_0 es constante (por simplicidad cero), por tanto usando (1.27) cero para todo tiempo

$$\Delta_{x,y,z} \psi = 0. \quad (1.28)$$

Por otro lado, asumiendo que la superficie de la tierra es plana, la velocidad u es horizontal y puesto que la presión se puede considerar como la función corriente se tiene que

$$u = (u_1, u_2) = \nabla^\perp \psi, \quad (1.29)$$

donde $\nabla^\perp = (-\partial_y, \partial_x)$. Usando la aproximación de Boussineq, podemos ver que la derivada vertical de ψ es proporcional a la temperatura potencial

$$\theta \propto \frac{\partial \psi}{\partial z}. \quad (1.30)$$

Además por la ley de la conservación de la energía tenemos que

$$\partial_t \theta + u \cdot \nabla \theta = 0. \quad (1.31)$$

Por (1.28), ψ satisface la ecuación de Laplace y usando el mapa de Dirichlet-Neumann

$$\theta = \frac{\partial \psi}{\partial z} = (-\Delta_{x,y})^{\frac{1}{2}} \psi, \quad (1.32)$$

donde $(-\Delta_{x,y})^{\frac{1}{2}}$ es el laplaciano fraccionario. Por tanto, (1.29), (1.31) y (1.32) nos permiten describir el siguiente sistema

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp \psi, \\ (-\Delta)^{\frac{1}{2}} \psi = \theta, \end{cases}$$

conocido como **ecuación cuasi-geostrófica superficial (SQG)**.

Como hemos visto este sistema se deriva a partir de la ecuación cuasi-geostrófica cuando hacemos ciertas simplificaciones en la geometría y aproximamos ciertas cantidades. Es claro que el objetivo principal de este modelo no es obtener una descripción exacta del fenómeno, sino una intuición y comprensión lo más completa del mismo. Esta ecuación no local de gran importancia geofísica describe la evolución de una superficie de temperatura en un fluido de vorticidad potencial estratificado de rotación rápida [Ped87; Con+94; Hel+95]. Podemos reescribir la ecuación de forma más compacta como

$$(SQG) : \begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0 \\ u = R^\perp \theta, \end{cases}$$

ya que el campo de velocidades $u(x, t)$ viene dado por

$$u = \nabla^\perp \psi = \nabla^\perp (-\Delta)^{-\frac{1}{2}} \theta = R^\perp \theta = (-R_2 \theta, R_1 \theta),$$

donde R_j son las transformadas de Riesz para $j = 1, 2$ y $\theta(x, t)$ es la temperatura potencial con $x \in \mathbb{R}^2$, $t \geq 0$. Más allá de su propio interés físico, la SQG tiene un gran interés matemático debido a la estrecha conexión que comparte con la ecuación 3D de Euler, [Con+94]. Para un fluido incompresible, 3D Euler, viene dado por

$$(E) : \begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p, \\ \operatorname{div} u = 0 \end{cases}$$

donde $u = (u_1, u_2, u_3)$ es el campo de velocidades del fluido y $p = (p_1, p_2, p_3)$ la presión. Tomando el rotacional a la ecuación anterior, se tiene que

$$(E_\omega) : \begin{cases} \partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u, \\ \omega = \nabla \times u, \end{cases}$$

conocida como ecuación de Euler en forma de vorticidad. Por otro lado, tomando el gradiente perpendicular en (SQG):

$$(SQG_{\nabla^\perp}) : \begin{cases} \partial_t \nabla^\perp \theta + u \cdot \nabla \nabla^\perp \theta = \nabla^\perp \theta \cdot \nabla u, \\ \operatorname{div} u = 0. \end{cases}$$

Nótese ahora, que el papel que juega la vorticidad ω en 3D Euler y el gradiente perpendicular de la temperatura potencial $\nabla^\perp \theta$, es exáctamente el mismo. En ambos casos, el transporte se ve afectado por una interacción cuadrática no lineal: $\omega \cdot \nabla u$ en (E_ω) y $\nabla^\perp \theta \cdot \nabla u$ en la (SQG_{∇^\perp}) . El gradiente de la velocidad ∇u está relacionado con ω y $\nabla^\perp \theta$, mediante un operador singular cuyo símbolo de Fourier es de orden cero. Estas analogías analíticas y geométricas (cf. [Con+94]) entre ambas ecuaciones, sugieren que un buen conocimiento del comportamiento de las soluciones de la SQG, pueda arrojar algo de luz para entender el caso de Euler tres dimensional.

En un primer lugar, las simulaciones numéricas presentaron ciertas evidencias donde se observaba la posible formación de frentes singulares [Con+94; OY97; Con+98]. Más tarde, estos posibles escenarios singulares fueron estudiados y descartados analíticamente [C98; CF01; CF02; FR11]. A pesar del gran esfuerzo de la comunidad matemática por avanzar en esta dirección, el problema de existencia global de soluciones suaves o la existencia de singularidades a tiempo finito para la SQG sigue abierto.

Otro modelo a tener en cuenta, es la ecuación cuasi-geostrófica superficial disipativa, dada por

$$(SQG_\alpha) : \begin{cases} \partial_t \theta + u \cdot \nabla \theta + \nu (-\Delta)^\alpha \theta = 0, \\ u = R^\perp \theta, \end{cases}$$

donde $\nu > 0$ es el coeficiente de viscosidad y $\alpha \in [0, 1]$. El operador no local $(-\Delta)^\alpha$ se define a través de la transformada de Fourier

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \hat{f}(\xi),$$

donde \hat{f} es la transformada de Fourier de f . Según el valor del parámetro α podemos distinguir tres casos: el caso supercrítico $0 \leq \alpha < \frac{1}{2}$, el caso crítico $\alpha = \frac{1}{2}$ y el caso subcrítico $\frac{1}{2} < \alpha \leq 1$. La criticidad se puede entender como una lucha entre el término no lineal y disipativo. Tomando transformada de Fourier en espacio en (SQG_α) , se tiene que

$$\partial_t \hat{\theta}(l, t) + \nu |l|^{2\alpha} \hat{\theta}(l, t) = - \sum_{j+k=l} \frac{j^\perp \cdot k}{|j|} \hat{\theta}(j, t) \hat{\theta}(k, t). \quad (1.33)$$

Ahora es fácil ver que el término no lineal y el término disipativo son comparables cuando $\alpha = \frac{1}{2}$. Sin embargo, cuando $\alpha > \frac{1}{2}$ el término disipativo gana a la no linealidad y cuando $\alpha < \frac{1}{2}$ la no linealidad gana al término disipativo. El problema sobre la existencia global y el estudio de singularidades a tiempo finito para la $(SQG)_\alpha$, ha sido muy estudiado y analizado en los últimos veinte años.

El caso subcrítico ($\alpha < \frac{1}{2}$) está entendido y esencialmente resuelto en [Res95; CW99], donde se prueba la existencia global de soluciones suaves mediante métodos

perturbativos y estimaciones de energía. Véase también [Wu01; Ju04; DL08], donde se estudian propiedades cualitativas de soluciones débiles y fuertes para el caso subcrítico.

El caso crítico ($\alpha = 1$) es mucho más desafiante debido al equilibrio entre el término no lineal y la disipación. Por tanto, los métodos perturbativos no son útiles; se necesitan nuevas estrategias más refinadas para atacar el problema. Veamos con algo más de detalle las técnicas que se desarrollaron, pues parte de esta memoria trata de estudiar y analizar las soluciones de esta ecuación en espacios ambientes más complejos que el euclídeo. El primer resultado de existencia global para la ecuación crítica probado en [Con+01], utilizaba como hipótesis la pequeñez del dato inicial. En particular, probaron que si $\|\theta_0\|_{L^\infty}$ es suficientemente pequeño, entonces para $\theta_0 \in H^1$ la solución permanece en H^1 globalmente. Para el mismo tipo de resultado con dato pequeño en otros espacios críticos, véase [CL03; CC04; Miu06; HK07].

De forma independiente, y con enfoques distintos en [Kis+07] y [CV10a], probaron la existencia global de soluciones para datos iniciales arbitrariamente grandes. La primera prueba en [Kis+07] es un resultado de propagación de regularidad. La idea se basa en construir una cierta familia de módulos de continuidad, de forma que si el dato cumple este módulo, la evolución de la ecuación lo preserva para todo tiempo. Esta técnica ha sido utilizada para resolver varias variantes del problema: en presencia de una fuerza externa [Fri+09], un término lineal dispersivo [KN10], ecuaciones de reacción-difusión generales [SV12] y generalizaciones singulares de la ecuación, etc.

Por otro lado, la prueba de [CV10a] se basa en las ideas usadas por E. De Giorgi para resolver el decimonoveno problema de Hilbert [DG57], adaptadas al carácter parabólico y no local de la ecuación. Más precisamente, prueban que cualquier solución débil se vuelve instantáneamente Hölder continua y por tanto regular para todo tiempo. La generalidad de la prueba permite que se aplique a ecuaciones de transporte generales en cualquier dimensión donde la velocidad $u \in L_t^\infty BMO_x$. Véase también [CV10b; Caf+11; Sil12; Sil+13] para aplicaciones del esquema de De Giorgi a otras ecuaciones de transporte-difusión.

En [KN09], los autores proponen una tercera prueba distinta a las anteriores, usando funciones test especiales para controlar la evolución de la norma Hölder.

Por último, una cuarta y definitiva prueba fue introducida por [CV12], usando cotas inferiores no lineales para los operadores no locales como el Laplaciano fraccionario. Las desigualdades ponen de manifiesto de forma cuantitativa el dominio de la disipación frente al término no lineal, las cuales no son visiblemente aparentes en las otras pruebas. Para que la prueba funcione, necesitan imponer una propiedad en el dato inicial denominada “*only small shocks (OSS)*” y controlar su degeneración

para todo tiempo. Una variante de esta prueba [Con+15], permite reemplazar la condicion de (OSS) usando el método de De Giorgi.

Por el momento, en el caso supercrítico ($\alpha > \frac{1}{2}$), el problema de existencia global de soluciones suaves o de singularidades a tiempo finito está completamente abierto. Sólomente se conoce existencia global en el caso de datos pequeños [CL03; CC04; Wu04; Miu06; Yu08] y la regularidad eventual de las soluciones débiles, [Sil10; Dab11].

La ecuación cuasi-geostrófica superficial crítica en la esfera dos dimensional

En la derivación de la SQG se han usado varias aproximaciones y simplificaciones de las ecuaciones cuasi-geostróficas. Por ejemplo, suponemos que superficie de la Tierra es plana, es decir, sin curvatura. En el Capítulo 2 de esta tesis tratamos de entender el comportamiento de las soluciones de la ecuación superficial crítica cuando el espacio ambiente tiene curvatura. Por tanto, de forma general, consideramos la SQG en una superficie compacta orientable M con métrica riemanniana g dada por

$$(SQG_M) : \begin{cases} \partial_t \theta + u \cdot \nabla_g \theta + \Lambda_g \theta = 0, \\ u = \nabla_g^\perp \Lambda_g^{-1} \theta = R_g^\perp \theta, \end{cases} \quad (1.34)$$

donde Λ_g es la raíz cuadrada del operador de Laplace-Beltrami $-\Delta_g$. Nuestro principal resultado en este capítulo, se centra en el caso físicamente más relevante,⁷ es decir, en la esfera dos dimensional:

Theorem 1.9 ([AO+18a]). *Sea $\theta_0 \in L^2(\mathbb{S}^2)$ y θ una solución débil del problema de Cauchy*

$$(SQG_{\mathbb{S}^2}) : \begin{cases} \partial_t \theta + u \cdot \nabla_g \theta = -\Lambda_g \theta, \\ \theta(x, 0) = \theta_0, \end{cases} \quad (1.35)$$

donde $u = \nabla_g^\perp \Lambda_g^{-1} \theta$. Entonces $\theta(x, t)$ es continua con un cierto módulo explícito de continuidad para todo $t > 0$.

En el caso particular de la esfera dos dimensional es sencillo computar explícitamente que $\text{div}_g u = 0$ y por tanto el fluido es incompresible. Esto es cierto también en caso de cualquier variedad riemanniana de dimensión dos. Su análogo en dimensiones mayores es más delicado. Cuando la dimensión es par y en presencia de estructura simpléctica, se puede construir canónicamente un gradiente ortogonal cuya divergencia es cero.

La idea de la demostración sigue en cierta medida la estrategia introducida en [CV10a], usando una versión no local y paraólica del método de De Giorgi. Como

⁷Naturalmente, motivado por entender la ecuación en la superficie de la Tierra

comentamos anteriormente, la generalidad de su prueba permite usarla en ecuaciones de transporte generales donde el campo de velocidades tiene oscilación media acotada (BMO). Sin embargo, nuestra situación no es tan propicia, pues la curvatura debe tenerse en cuenta. Además, puesto que no podemos hacer reescalamientos en nuestra ecuación, a diferencia del caso euclídeo, nuestra exposición hace especial énfasis en tener en cuenta la influencia de las escalas en nuestros argumentos.

Grosso modo el esquema de De Giorgi, o la variante que se usa para este problema, trata de mejorar la regularidad de la solución de L^2 a C^α , cuya norma viene dada por

$$\|f\|_{C^\alpha} := \|f\|_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_g(x, y)^\alpha}.$$

Siguiendo esta estrategia, comenzamos probando la existencia global de soluciones débiles de (SQG_M) para datos iniciales $\theta_0 \in L^2_x$; véase Apéndice A. El siguiente paso consiste en mejorar esta regularidad pasando de L^2_x a L^∞_x . Para ello probamos cotas uniformes del supremo esencial en espacio y tiempos positivos $t \geq t_0 > 0$, i.e.

$$|\theta(x, t)| \leq C(M, t_0, \|\theta_0\|_{L^2(M)})$$

para todo $x \in M$ y $t \geq t_0 > 0$. Como se puede ver en la Sección 2.1, esta prueba se basa en conseguir una recurrencia no lineal entre energías truncadas. Por tanto, en este punto sabemos que nuestra solución $\theta \in L_t^\infty L_x^\infty$, lo que nos lleva a que $u = R_g^\perp \theta \in L_t^\infty L_x^p$ para todo $p \in [2, \infty)$. Esto nos permite despreciar la dependencia entre u y θ , y así considerar el problema como uno de reacción-difusión donde la velocidad u está en un cierto espacio funcional.

En la Sección 2.2, demostramos dos lemas técnicos dedicados a la obtención de principios del máximo para una familia de barreras adaptadas a nuestra geometría. Estas jugarán un papel esencial en el paso de las cotas L_x^∞ a C_x^α . En realidad, cuando queramos demostrar regularidad C_x^α , lo que haremos es ver que hay un decaimiento de la oscilación de θ , lo cual implicará la cota C_x^α directamente.⁸

En la siguiente Sección 2.3, presentamos una desigualdad local de energía que, junto a las barreras de la sección anterior, proveen la recurrencia no lineal adecuada que permite probar un decaimiento de la oscilación de θ . Para demostrar esta desigualdad (c.f. Lema 2.3) necesitamos imponer una condición extra en la velocidad u que no se cumple para la (SQG_M) . Por lo tanto llegados a este punto, el resto del argumento (véase Sección 2.4) nos permitirá demostrar el decaimiento de la oscilación, proporcionando el siguiente teorema:

⁸Este argumento es bien conocido, por lo tanto no lo incluiremos en la introducción para facilitar su lectura.

Theorem 1.10 ([AO+18a]). Sea $\theta_0 \in L^2(M)$ y θ una solución a débil del problema de Cauchy

$$\begin{cases} \partial_t \theta + u \cdot \nabla_g \theta = -\Lambda_g \theta \\ \theta(x, 0) = \theta_0 \end{cases}$$

donde el campo de velocidades incompresible $u \in L^\infty(M)$ uniformemente en tiempo. Entonces $\theta(x, t)$ es de clase C^α para cualquier $t > 0$.

Podemos pensar en este resultado, como una versión subcrítica del problema, pues $u \in L^\infty(M)$ se satisface por ejemplo para velocidades del tipo $u = \nabla_g^\perp \Lambda_g^{-1-\epsilon} \theta$ para cualquier $\epsilon > 0$. En el caso euclídeo y sin la condición de incompresibilidad, este resultado ha sido probado en [Sil12]. La prueba se basa en principios de comparación cuantitativos, difíciles de adaptar en nuestro contexto.

Esta Sección 2.4 se divide en tres partes. En primer lugar, en la Subsección 2.4.1, probaremos el decaimiento de la oscilación de θ bajo una hipótesis de pequeñez del dato inicial (cf. Proposición 2.2). En la Subsección 2.4.2, quitamos la hipótesis de dato pequeño usando una versión no local de la desigualdad isoperimétrica de De Giorgi (cf. Lema 2.6). En la última Subsección 2.4.3, unimos las piezas anteriores para probar el Teorema 1.10.

Finalmente en la Sección 2.5, demostramos el Teorema 1.9 modificando los argumentos usados anteriormente para probar el Teorema 1.10, donde la velocidad no satisface a priori la hipótesis de la desigualdad de energía local (cf. (2.3)). Para ello, adaptamos con un argumento geométrico la estrategia de [CV10a], donde se apoyan en los reescalamientos y las traslaciones. En nuestro caso, usando el grupo de rotaciones de la esfera, somos capaces de probar un módulo de continuidad logarítmico.

Existencia global de soluciones fuertes para la ecuación cuasi-gestrófica superficial crítica

En el tercer capítulo de la tesis continuamos con el estudio de la ecuación cuasi-gestrófica superficial (1.34), pero en este caso nos interesamos en la existencia global de soluciones en espacios de Sobolev. Concretamente probamos el siguiente resultado:

Theorem 1.11 ([AO+18b]). Dado un dato inicial $\theta_0 \in H^s(\mathbb{S}^2)$ con $s > \frac{3}{2}$, existe una solución global $\theta(x, t)$ de $(SQG_{\mathbb{S}^2})$ en $H^s(\mathbb{S}^2)$. Además, la solución se vuelve instantáneamente regular.

Para la demostración de este teorema necesitaremos tres ingredientes básicos: primero una representación integral del operador Laplace-Beltrami fraccionario

en la esfera.⁹Segundo, tendremos que adaptar la estrategia introducida en [CV12] basada en principios del máximo no locales para el laplaciano fraccionario. Por último utilizaremos el Teorema 1.9, el cual nos proporciona un módulo de continuidad, que usaremos como sustituto de la condición de *only small shocks* en [CV12].

La estructura de la prueba se divide en la siguiente forma:

En la Sección 3.1 se presentan las herramientas necesarias para el resto del capítulo. Entre ellas la representación integral que hemos demostrado en [AO+18c], crucial en esta demostración, y la descripción del sistema de coordenadas adecuados que usaremos, basada en las rotaciones y la proyección estereográfica (véase Figura (3.1)). Este sistema de coordenadas nos permitirá, probar ciertas estimadas y conmutadores de operadores no locales (cf. Lemma 3.1).

Con estas herramientas en mano, en la Sección 3.2, probamos una cota inferior no lineal para el operador Laplace-Beltrami fraccionario siguiendo la estrategia en [CV12]. Pero en nuestro caso tenemos que lidiar con las dificultades añadidas que suponen los términos de error debidos a los efectos de la curvatura de la esfera.

Por último, en la Sección 3.3, nuestro objetivo es probar una cota global en tiempo para el gradiente de nuestra solución en L^∞ , véase Proposición 3.2. Para ello realizamos estimaciones a priori en el gradiente de la velocidad utilizando los resultados anteriores y ayudándonos del Teorema 1.9. Como se puede ver en el Apéndice B, esta proposición implica directamente la existencia global de soluciones en espacio de Sobolev concluyendo la prueba del Teorema 1.11.

Modelos tipo *slice*

Dentro del área de las ciencias atmosféricas, además de la ecuación cuasi-geostrófica superficial existen otros modelos, como son los modelos tipo "*slice*". Estos se usan frecuentemente en el estudio del comportamiento climático, más concretamente en la formación de frentes atmosféricos cuyas predicciones son esenciales en meteorología [Cul07; Vis14].

En [CH13], Cotter y Holm introdujeron un modelo tipo *slice* conocido como *Cotter-Holm-Slice-Model* (CHSM) que se usa para estudiar el movimiento oceánico y atmosférico que tienen lugar en un corte (*slice*) vertical $\Omega \subset \mathbb{R}^2$. El movimiento del fluido en este corte vertical se acopla dinámicamente a la velocidad del flujo transversal al corte, que se supone que varía linealmente con la distancia normal a este. Esta suposición en el CHSM hace que sea un modelo más simplificado sin dejar de captar los aspectos importantes de un flujo tridimensional.

⁹Este resultado de interés independiente, en colaboración con A. Córdoba y A.D. Martínez en [AO+18c], conforma parte de la tesis de este último.

En [HB71], se pueden encontrar otros modelos atmosféricos estudiando la frontogénesis que dan otro tipo de aproximación matemática de los frentes, proporcionando una teoría general para su estudio.

Si hablamos de la formación de frentes no podemos olvidarnos de la inestabilidad baroclínica, estudiada por Eady en un modelo clásico de 1949 [Ead49]. Numerosos estudios sobre esta inestabilidad, han concluido que esta es una de las fuentes más importantes de variaciones de la escala sinóptica en la atmósfera. En [BAWiHoFe], se puede ver que esto está directamente vinculado con sistemas de frentes, como por ejemplo, en la formación de remolinos en el Mar del Norte.

Incompressible Slice Model

En el Capítulo 4 estudiamos un caso particular del CHSM conocido como *Incompressible Slice Model* (ISM)¹⁰. Se debe enfatizar que el ISM es potencialmente útil en simulaciones numéricas de frentes. Por ejemplo, dado que el dominio es un corte bidimensional, las simulaciones tardan mucho menos tiempo en ejecutarse que en un modelo tridimensional completo. Se han realizado muchos estudios sobre este tipo de modelos idealizados para predecir y examinar la formación y evolución de los frentes climáticos (cf. [NH89; Bud+13; Vis14; Vis+14; Yam+17]).

Las ecuaciones de evolución para este modelo con campo de velocidades $u_S(x, z, t) : \Omega \subset \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$, el escalar transversal al corte $u_T(x, z, t) : \Omega \subset \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}$, como también la temperatura potencial $\theta_S(x, z, t) : \Omega \subset \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}$, están dadas por

$$\partial_t u_S + u_S \cdot \nabla u_S - f u_T \hat{x} = -\nabla p + \frac{g}{\theta_0} \theta_S \hat{z}, \quad (1.36)$$

$$\partial_t u_T + u_S \cdot \nabla u_T + f u_S \cdot \hat{x} = -\frac{g}{\theta_0} z s, \quad (1.37)$$

$$\partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T s = 0, \quad (1.38)$$

$$\nabla \cdot u_S = 0. \quad (1.39)$$

Aquí g representa la aceleración debido a la gravedad, θ_0 es la temperatura potencial de referencia, f la fuerza de Coriolis, la cual se asume constante, y s es otra constante que mide la variación de la temperatura potencial en la dirección transversal. En estas ecuaciones, ∇ denota el gradiente 2D en el corte, p es la presión obtenida por la incompresibilidad del fluido en el corte ($\nabla \cdot u_S = 0$), mientras que \hat{x} y \hat{z} denotan los vectores horizontales y verticales unitarios respectivamente. Además consideraremos como condición de contorno

$$u_S \cdot n = 0 \text{ on } \partial\Omega, \quad (1.40)$$

¹⁰Este modelo también es conocido en la literatura con el nombre de *Euler-Boussinesq Eady model*

donde n representa el vector normal unitario exterior a la frontera $\partial\Omega$.

El ISM comparte cierta semejanza con las ecuaciones de 2D Boussinesq, que se usan comúnmente para modelar flujos atmosféricos y oceánicos a gran escala responsables de los frentes fríos y las corrientes en chorro [Ped87]. Nótese que podemos recuperar el sistema 2D Boussinesq haciendo $f, u_T = 0$. La variable u_T representa la velocidad transversal en el corte, la cual aparece de manera natural cuando derivamos las ecuaciones, dando lugar a un sistema de ecuaciones acoplado más complejo. Las ecuaciones de Boussinesq se han estudiado ampliamente prestando especial interés en los resultados de existencia y regularidad [CD80; Cha06; HL05]. Sin embargo, la pregunta fundamental sobre la existencia global o singularidades a tiempo finito permanece abierta. Este problema incluso se discute en *Eleven great problems of mathematical hydrodynamics*, [Yud03]. Varios avances importantes en el problema de regularidad global, basados en evidencias numéricas, proveen que las soluciones de las ecuaciones de 3D Euler con simetría axial, que pueden identificarse con la ecuación inviscida de 2D Boussinesq, desarrollan singularidades en tiempo finito cuando el dominio tiene un límite sólido, [LH14b; LH14a]. Recientemente, en [EJ18] han probado la formación de singularidades a tiempo finito para soluciones del sistema 2D Boussinesq cuando el dominio es un sector de ángulo menor a π .

El objetivo de este capítulo consta de tres partes:

En primer lugar caracterizamos una clase particular de soluciones de equilibrio del ISM, y estudiamos la estabilidad formal y no lineal del modelo, dando lugar a los siguientes teoremas:

Theorem 1.12 ([AOL18b]). *Una clase de soluciones estacionarias del ISM (1.36)-(1.39) con condición de contorno (1.40) viene dada por los puntos críticos del hamiltoniano generalizado*

$$H_\Phi = \int_\Omega \left\{ \frac{1}{2}(|u_S|^2 + u_T^2) - \gamma_S \theta_S \right\} dV + \int_\Omega \Phi(q) dV + \sum_{i=0}^n a_i \int_{\partial\Omega_i} v_S \cdot ds.$$

Están dadas por las condiciones

$$a_i = \Phi'(q_e|_{\partial\Omega_i}), \quad \text{for } i = 0, \dots, n, \quad (1.41)$$

$$u_{Se} = -\text{curl}(\Phi'(q_e)\hat{z})_s, \quad (1.42)$$

$$u_{Te} = \text{curl}(\Phi'(q_e)\hat{z}) \cdot \nabla \theta_{Se}, \quad (1.43)$$

$$\gamma_S = \text{curl}(\Phi'(q_e)\hat{z}) \cdot (\nabla u_{Te} + f\hat{x}). \quad (1.44)$$

Aquí $\gamma_S = (g/\theta_0)z$, $v_S = su_S - (u_T + fx)\nabla\theta_S$ es la velocidad de circulación en el (ISM) y $q = \text{curl}(v_S) \cdot \hat{z}$ es la vorticidad potencial. Además, Φ se puede escribir en términos de la función de Bernoulli K para la solución estacionaria

$$\Phi(\lambda) = \lambda \left(\int_{\lambda} \frac{K(t)}{t^2} dt + C \right).$$

Theorem 1.13 ([AOL18b]). *Un punto de equilibrio del (ISM) perteneciente a la clase especificada en el Teorema 1.12 es formalmente estable si*

$$\frac{(\hat{z} \times \nabla q_e) \cdot u_{Se}}{|\nabla q_e|^2} > 0. \quad (1.45)$$

Este último resultado se asemeja al primer Teorema de estabilidad de Arnold para la ecuación de 2D Euler incompresible [Arn89], donde la condición de estabilidad viene dada por

$$\frac{(\hat{z} \times \nabla \omega_e) \cdot u_e}{|\nabla \omega_e|^2} > 0.$$

Aquí $\omega_e = \text{curl}(u_e) \cdot \hat{z}$. El término extrat $q_R = -\text{curl}((u_T + fx)\nabla\theta_S)$ que aparece en (1.45) se debe al término de velocidad transversal u_T y temperatura potencial θ_S .

Theorem 1.14 ([AOL18b]). *Podemos definir una norma Q en $\mathcal{X}(\Omega) \widehat{\otimes} \mathcal{F}(\Omega) \times \wedge^2(\Omega)$ tal que una solución de equilibrio del (ISM) perteneciente la clase específica del Teorema 1.12 es no linealmente estable con respecto a Q si*

$$0 < \lambda_1 \leq \frac{(\hat{z} \times \nabla q_e) \cdot u_{Se}}{|\nabla q_e|^2} \leq \lambda_2 < \infty.$$

En segundo lugar probamos la existencia local de soluciones del (ISM) en espacios de Sobolev:

Theorem 1.15 ([AOL18b]). *Para $s > 2$ entero y dato inicial $(u_S^0, u_T^0, \theta_S^0) \in H_*^s(\Omega) \times H^s(\Omega) \times H^s(\Omega)$, existe un tiempo $T = T(\|(u_S^0, u_T^0, \theta_S^0)\|_{H^s}) > 0$ tal que las ecuaciones del (ISM) (1.36)-(1.39) con condición de contorno (1.40) tiene una única solución (u_S, u_T, θ_S) en $C([0, T]; H_*^s(\Omega) \times H^s(\Omega) \times H^s(\Omega))$.*

Por último también probamos un criterio de continuidad de soluciones:

Theorem 1.16 ([AOL18b]). *Supongamos que $(u_S^0, u_T^0, \theta_S^0) \in H_*^s(\Omega) \times H^s(\Omega) \times H^s(\Omega)$ para $s > 2$ entero y que la solución (u_S, u_T, θ_S) de las ecuaciones (1.36)-(1.39) con condición de contorno (1.40) es de clase $C([0, T]; H^s(\Omega) \times H^s(\Omega) \times H^s(\Omega))$. Entonces para $T^* < \infty$, las siguientes dos condiciones son equivalentes:*

$$(i) \quad E(t) < \infty, \forall t < T^* \text{ y } \limsup_{t \rightarrow T^*} E(t) = \infty, \quad (1.46)$$

$$(ii) \quad \int_0^t \|\nabla u_S(s)\|_{L^\infty} ds < \infty, \forall t < T^* \text{ y } \int_0^{T^*} \|\nabla u_S(s)\|_{L^\infty} ds = \infty, \quad (1.47)$$

donde $E(t) = \|u_S\|_{H^s}^2 + \|u_T\|_{H^s}^2 + \|\theta_S\|_{H^s}^2$. Si T^* existe entonces T^* se le conoce como el primer tiempo de explosión y (1.47) es un criterio de explosión o continuidad.

La estructura del capítulo será la siguiente:

En la Sección 4.1 introducimos algunas definiciones básicas y lemas conocidos sobre los espacios de Sobolev. También incluimos varios resultados preliminares conocidos como el Teorema de estabilidad de Arnold y el Teorema de Kato-Lai para ecuaciones de evolución no lineales. La Sección 4.2 introduce el CHSM utilizando su formulación lagrangiana, como se lleva a cabo en [CH13]. En particular, en la Subsección 4.2.3 sustituimos el lagrangiano de Euler-Boussinesq y derivamos las ecuaciones de ISM en la cual se centra nuestro estudio. Para acabar, en la Subsección 4.2.4 introducimos algunas cantidades conservadas, que serán fundamentales para estudiar la estabilidad del sistema.

En la Sección 4.3, estudiamos la estabilidad de soluciones del ISM, usando el algoritmo Energy-Casimir [Hol+85]. Para ello en primer lugar, caracterizamos una clase de soluciones de equilibrio del ISM en la Subsección 4.11. A continuación en la subsecciones 4.3.2 y 4.3.3 estudiamos la estabilidad formal y no lineal a su alrededor de la soluciones de equilibrio.

En la Sección 4.4 proporcionamos el resultado de existencia local de soluciones del ISM adaptando un resultado abstracto para sistemas de ecuaciones no lineales [KL84]. Antes de comenzar con la prueba, probamos ciertas estimadas y hacemos algunos observaciones previas. Finalmente, en la Subsección 4.4.1 probamos la existencia y unicidad de soluciones, demostrando el Teorema 1.15.

En la Sección 4.5, construimos un criterio de continuación del ISM, usando estimaciones de energía, probando el Teorema 1.16.

Comentarios finales y conclusiones

Esta tesis se centra en el análisis de ecuaciones en derivadas parciales no lineales que tratan de explicar algunos fenómenos geofísicos y de la mecánica de fluidos. En particular, nos centramos en el estudio de dos modelos. Por un lado, probamos resultados de existencia y regularidad global de soluciones de la ecuación cuasi-geostrófica superficial crítica en la esfera. Por otro lado, estudiamos el comportamiento de las soluciones del *Incompressible Slice Model*, caracterizando las soluciones estacionarias, probando la existencia local y estableciendo un criterio de continuidad.

Todos estos resultados se recogen en la siguiente lista de publicaciones:

- D. Alonso-Orán, A. Córdoba and A. D. Martínez, *Continuity of weak solutions of the critical surface quasigeostrophic equation on S^2* , *Advances in Mathematics*, **328** (2018), pp. 264–299. ISSN 0001-8708,

<https://doi.org/10.1016/j.aim.2018.01.015>.

(Capítulo 2)

- D. Alonso-Orán, A. Córdoba and A. D. Martínez, *Global well-posedness of critical surface quasigeostrophic equation on the sphere*, Advances in Mathematics, **328** (2018), pp. 248–263. ISSN 0001-8708, <https://doi.org/10.1016/j.aim.2018.01.016>. (Capítulo 3)
- D. Alonso-Orán and A. Bethancourt de León, *Stability, well-posedness and blow-up criterion for the Incompressible Slice Model*, Physica D: Nonlinear Phenomena, 2018, ISSN 0167-2789, <https://doi.org/10.1016/j.physd.2018.12.005>. (Capítulo 4)

Además en el Apéndice C, describo brevemente otros resultados que he obtenido durante el transcurso de mi doctorado. Estos no han sido añadidos al núcleo de esta tesis pues no se encuadran dentro del tema principal. La lista de artículos que los conforman es la siguiente:

- D. Alonso-Orán and A. Bethancourt de León and S. Takao, *The Burgers' equation with stochastic transport: shock formation, local and global existence of smooth solutions*, arXiv:1808.07821, 2018.
- D. Alonso-Orán, F. Chamizo, A. D. Martínez and A. Mas, *Pointwise monotonicity of heat kernels*, arXiv:1807.11072, 2018.
- D. Alonso-Orán and A. Bethancourt de León, *On the well-posedness of stochastic Boussinesq equations with cylindrical multiplicative noise*, arXiv:1807.09493, 2018.

Continuity of weak solutions for the critical SQG on the sphere

The aim of this chapter is to show the continuity of weak solutions to the surface quasi-geostrophic equation on the two dimensional sphere stated in Theorem 1.1. Recall that the equation is given by

$$(SQG_M) : \begin{cases} \partial_t \theta + u \cdot \nabla_g \theta + \Lambda_g \theta = 0, \\ u = \nabla_g^\perp \Lambda_g^{-1} \theta = R_g^\perp \theta, \end{cases} \quad (2.1)$$

where $\Lambda_g = (-\Delta_g)^{\frac{1}{2}}$ is the square root of the Laplace-Beltrami operator. As mentioned in the introduction, the proof is subdivided into several sections, each of them taking care of a specific part of the argument. Our final goal is to give a proof to Theorem 1.1, but as explained before, this will follow from Theorem 1.2 using a geometrical twist. Therefore, since Theorem 1.2 holds for a compact orientable manifold (M, g) , we will prove it in this general setting. This will also emphasize the point where we need to restrict our approach to the two dimensional sphere. Without any further ado, let us begin with the first step of the discussion.

2.1 $L_{x,t}^\infty$ bound

In this section we illustrate De Giorgi's method which will be based on a non linear inequality for some sort of energy. A finer and subtle version of this but with the same flavor, will be exposed in Section 2.4.

Proposition 2.1. *Let $\theta(x, t)$ be a weak solution of (2.1) (cf. Appendix A). Then for any fixed $t_0 > 0$ there exists a positive constant C that depends only on $\|\theta_0\|_{L^2}$, t_0 and the manifold M such that*

$$|\theta(x, t)| \leq C \text{ for any } x \in M \text{ and any } t > t_0.$$

Remark 2.1. *In the rest of this chapter all constants C will be assumed to depend implicitly on quantities that are considered fixed. In particular, they will have to be scale independent. Notice also that the constant might differ from line to line for the sake of the exposition's clearness.*

Proof of Proposition 2.1. We will proceed using a nonlinear energy inequality for consecutive energy truncations which is based on the interplay between a global

energy inequality and Sobolev inequality. Let us assume without loss of generality that $\int_M \theta(x, t) d\text{vol}_g(x) = 0$ and define the truncation levels

$$\ell_k = C(1 - 2^{-k}),$$

where C will be chosen later to be large enough. The k -th truncation of θ at the level ℓ_k will be denoted by $\theta_k = (\theta - \ell_k)_+$. Notice $(a)_+ = \max\{a, 0\}$ is a convex function. One can derive a differential inequality for the truncations using the Córdoba-Córdoba pointwise inequality for fractional powers of the Laplace-Beltrami operator on manifolds (cf. [CM15])

$$\partial_t \theta_k + u \cdot \nabla_g \theta_k \leq -\Lambda \theta_k.$$

Multiplying this by θ_k , integrating in M , using that u is divergence free and using the self-adjointness of the fractional Laplace-Beltrami operator the following holds

$$\partial_t \int_M \theta_k^2 d\text{vol}_g(x) + \int_M |\Lambda^{1/2} \theta_k|^2 d\text{vol}_g(x) \leq 0.$$

Let us introduce also truncation levels in time, namely, $T_k = t_0(1 - 2^{-k})$. Integrating this equation in time between s and t , where $s \in [T_{k-1}, T_k]$ and $t \in [T_k, \infty]$, yields

$$\int_M \theta_k^2(t) d\text{vol}_g(x) + 2 \int_s^t \int_M |\Lambda^{1/2} \theta_k|^2 d\text{vol}_g(x) dt \leq \int_M \theta_k^2(s) d\text{vol}_g(x).$$

Taking the supremum over $t \geq T_k$,

$$\sup_{t \geq T_k} \int_M \theta_k^2 d\text{vol}_g(x) + 2 \int_s^\infty \int_M |\Lambda^{1/2} \theta_k|^2 d\text{vol}_g(x) dt \leq \int_M \theta_k^2(s) d\text{vol}_g(x).$$

The right hand side dominates the following quantity

$$E_k := \sup_{t \geq T_k} \int_M \theta_k^2 d\text{vol}_g(x) + 2 \int_{T_k}^\infty \int_M |\Lambda^{1/2} \theta_k|^2 d\text{vol}_g(x) dt.$$

Taking the mean value on the resulting inequality on the interval

$s \in [T_{k-1}, T_k]$ gives

$$E_k \leq \frac{2^k}{t_0} \int_{T_{k-1}}^\infty \int_M \theta_k^2 d\text{vol}_g(x) dt.$$

Notice that for any $x \in M$ such that $\theta_k(x) > 0$ one also has, by construction, that $\theta_{k-1}(x) \geq 2^{-k}C$. Therefore,

$$\chi_{\{\theta_k > 0\}} \leq \left(\frac{2^k}{C} \theta_{k-1} \right)^{2/n}.$$

As a consequence of this

$$\begin{aligned} E_k &\leq \frac{2^k}{t_0} \int_{T_{k-1}}^{\infty} \int_M \theta_{k-1}^2 \chi_{\{\theta_k > 0\}} d\text{vol}_g(x) dt \\ &\leq \frac{2^{k(1+\frac{2}{n})}}{t_0 C^{2/n}} \int_{T_{k-1}}^{\infty} \int_M \theta_{k-1}^{2(n+1)/n} d\text{vol}_g(x) dt. \end{aligned}$$

Taking into account that E_{k-1} controls θ_{k-1} in $L_t^\infty L_x^2$ and the Sobolev embedding $L_t^2 H_x^{\frac{1}{2}} \hookrightarrow L_t^2 L_x^{2n/(n-1)}$, we can infer via Hölder's inequality that it also controls $L_{t,x}^{2(n+1)/n}$. Indeed, Sobolev and Poincaré inequalities yields

$$\begin{aligned} \int_{T_{k-1}}^{\infty} \int_M \theta_{k-1}^{2(n+1)/n} d\text{vol}_g(x) dt &\leq \int_{T_{k-1}}^{\infty} \left(\int_M \theta_{k-1}^2 d\text{vol}_g(x) \right)^{\frac{1}{n}} \left(\int_M \theta_{k-1}^{2n/(n-1)} d\text{vol}_g(x) \right)^{(n-1)/n} dt \\ &\leq 2E_{k-1}^{\frac{1}{n}} \int_{T_{k-1}}^{\infty} \int_M |\Lambda^{\frac{1}{2}} \theta_{k-1}|^2 d\text{vol}_g(x) dt \leq E_{k-1}^{1+1/n}. \end{aligned}$$

Therefore, we get the nonlinear recurrence

$$E_k \leq \frac{2^{k(1+\frac{2}{n})+1}}{t_0 C^{2/n}} E_{k-1}^{1+\frac{1}{n}},$$

which, for the sake of simplicity, can be rewritten as

$$E_k \leq C' 2^{k(1+2\epsilon)} E_{k-1}^{1+\epsilon},$$

where $C' = C'(M, t_0, n)$ and $\epsilon = \frac{1}{n}$. We claim that this sequence E_k converges to zero if C' is large enough (i.e. C' is small enough). Indeed, let us show by induction that $E_k \leq \delta^k E_0$ for $\delta = 2^{-(2\epsilon+1)\frac{1}{\epsilon}} < 1$, independent of k . First, we have that

$$\frac{E_k}{E_{k-1}} \leq C' 2^{k(1+2\epsilon)} E_{k-1}^\epsilon \leq \delta.$$

For $k = 1$, we easily get

$$C' 2^{(1+2\epsilon)} E_0^\epsilon \leq \delta$$

choosing the parameter C' sufficiently small. By the induction hypothesis

$$E_k \leq \delta E_{k-1} \text{ and } C' 2^{k(1+2\epsilon)} E_{k-1}^\epsilon \leq \delta.$$

Hence,

$$\begin{aligned} C' 2^{(k+1)(1+2\epsilon)} E_k^\epsilon &\leq C' 2^{(k+1)(1+2\epsilon)} (\delta E_{k-1})^\epsilon \\ &= C' 2^{k(2\epsilon+1)} E_{k-1}^\epsilon 2^{(2\epsilon+1)} \delta^\epsilon \leq \delta. \end{aligned}$$

Notice that the non-linear part disappears since u is divergence free. Furthermore one can mimic the proof for $-\theta$ to achieve the same bound for $|\theta|$. \square

2.2 Constructions of the barrier functions

In this section we provide constructions and properties of some barrier functions which are useful later in the argument. They represent one important part of the result. Lemmas 2.1 and 2.2 can be interpreted as a quantitative maximum principle for specific boundary elliptic problems at different scales.

To continue, we need to introduce the following notation: in this section we will work on a product space that corresponds to a space variable $x \in M$ times $z \in \mathbb{R}$, so $N = n + 1$ is its dimension. The arguments in the following sections deal with local properties around some fixed point $x_0 \in M$ and a geodesic ball around it $B_g(h)$ of radius h in the metric g . The dependence on the point is omitted since our conclusions are uniform due to the compactity of M . The usual euclidean metric will be denoted by $g = e$. We will deal with cylinders $(x, z) \in B_g^*(r, h) = B_g(r) \times I(h)$, where $I(h)$ denotes an interval in the variable z of length h . Usually, its endpoints will be irrelevant (in the few cases where they are relevant we will point it out explicitly). By a slight abuse of notation we will denote $B_g^*(h, h)$ by $B_g^*(h)$.

Lemma 2.1. *Let the function $b_1(x, z)$ (see Figure 2.1) satisfy*

$$\begin{cases} (\partial_z^2 + \Delta_g)b_1 = 0 & \text{in } B_g^*(h), \\ b_1 = 0 & \text{in } B_g(h) \times \partial I(h), \\ b_1 = 1 & \text{in } \partial B_g(h) \times I(h). \end{cases}$$

Then there exists a $\delta < 1$ independent of the scale h such that for any $x \in B_g(h/2) \times I(h)$

$$b_1(x, z) < \delta + O(h).$$

Proof of Lemma 2.1. Let $b(x, z)$ be the euclidean version at scale one

$$\begin{cases} (\partial_z^2 + \Delta_e)b = 0 & \text{in } B_e^*(1), \\ b = 0 & \text{in } B_e(1) \times \partial I(1), \\ b = 1 & \text{in } \partial B_e(1) \times I(1). \end{cases}$$

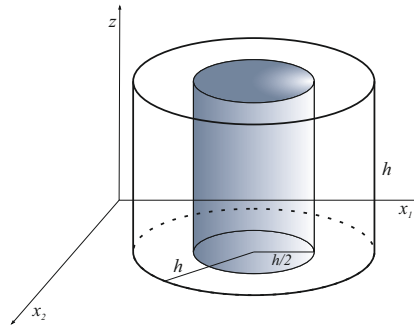


Fig. 2.1.: Barrier b_1

Then $b(x/h)$ satisfies the same equation at scale h . Define δ as the supremum of $b(x)$ in $B_e(1/2) \times I(1)$, which is strictly smaller than one by the maximum principle (cf. [PW84; Eva13]). We will treat b_1 as a perturbation of $b(x/h)$, the difference $u(x) = b_1(x) - b(x/h)$ satisfies

$$\begin{cases} (\partial_z^2 + \Delta_g)u = O(h^{-1}) \frac{\partial b}{\partial \rho} & \text{in } B_g^*(h), \\ 0 & \text{in } \partial B_g^*(h), \end{cases}$$

where ρ is the geodesic radius. Using Green's function for the geodesic problem we represent

$$u(x) = O(h^{-1}) \int_{B_g^*(h)} G_g(x, y) \frac{\partial b}{\partial \rho}(y/h) d\text{vol}_g(y).$$

The integral is bounded (up to a constant dependent on M) by

$$\int_{B_e^*(h)} \frac{1}{|x - y|^{N-2}} dy = O(h^2).$$

□

Remark 2.2. In the latter bound we used the fact that $G_g(x, y) = O(d(x, y)^{2-N})$ for $N \geq 3$ a fact that follows because the singularity is of that particular order and a maximum principle. The leading term in Hadamard's parametrix shows that the singularity has that prescribed order if $N \geq 3$ (cf. [HÖ7; Ste70; Fol95]). The constants involved depend continuously on the riemannian distortion of the euclidean metric which can be estimated uniformly due to the assumed compacity.

In the following we prove a variant of Lemma 2.1 we will exploit later, namely:

Lemma 2.2. Let $h \leq r$. There exist a function b_2 such that

$$\begin{cases} (\partial_z^2 + \Delta_g)b_2 = 0 & \text{in } B_g^*(r, h), \\ b_2 \geq 0 & \text{in } B_g(r) \times \partial I(h), \\ b_2 = 1 & \text{in } \partial B_g(r) \times I(h), \end{cases}$$

satisfying for $r_1 \leq r$

$$\sup_{x \in B_g^*(r_1, h)} b_2(x, z) \leq C \left\{ \frac{hr^{N-2}}{(r - r_1)^{N-1}} + r \right\}.$$

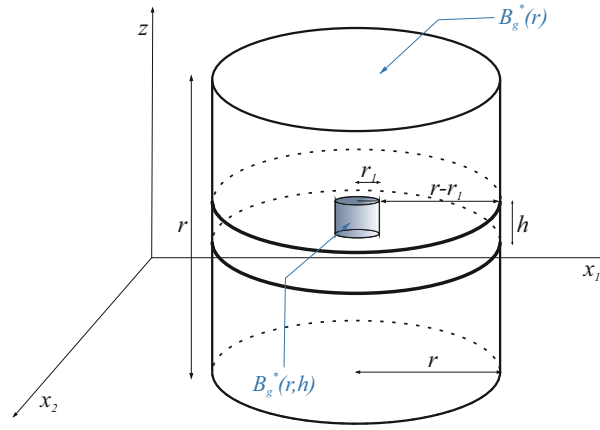


Fig. 2.2.: Barrier b_2

Proof of Lemma 2.2. We will use again a perturbation method argument: first the euclidean and later the general case controlling the difference. If the metric was the euclidean, we could invoke Green's function estimates yielding the result. Indeed, consider $B_e^*(r, h) \subseteq B_e^*(r)$ and let b be the restriction of a harmonic function in $B_e^*(r)$ (we are making the domain larger, see the Figure 2.2) with non negative boundary values defined to be equal to one near the equator and vanishing outside of it. The maximum principle assures that such a function is non negative and, by construction, satisfies all the assumptions. Finally, observe that integrating against the Poisson kernel $\partial_\nu G(x, y) = O(r^{-1}|x - y|^{1-N})$ provides the searched estimate. The need to make the domain larger allows us to rescale the Poisson kernel in the domain $B_e^*(1)$ which implies $\nabla G(x, y) = O(r^{-1}|x - y|^{1-N})$. Otherwise the constant involved might depend on the domain under consideration and, as a matter of fact, on the scale h (cf. [Wid67], one might in fact round the domain to make it C^2 if necessary without affecting the argument above.) Alternatively, one may rescale first to obtain the bound which is invariant upon rescaling and then rescale back (see Figure 2.2.)

As in our previous lemma, one treats $u = b_2 - b$ as a perturbation that satisfies the following boundary value problem:

$$\begin{cases} (\partial_z^2 + \Delta_e)u = k \frac{\partial b_2}{\partial \rho} & \text{in } B_g^*(r), \\ 0 & \text{in the boundary.} \end{cases}$$

Here k is a differentiable function independent of z of size $O(r)$ (cf. [Cha93], Theorem 2.17; which can be computed explicitly for the sphere).

Next we can estimate the derivative using Gauss' divergence theorem. To do that let $x = (\rho, \sigma, z)$ be the cylindrical geodesic coordinates: ρ geodesic distance, σ

angular direction, z the orthogonal variable. As before one has the following integral representation

$$u(x) = \int_{B_e^*(r)} G_e(x, y) k(y) \frac{\partial b_2(y)}{\partial \rho} dy.$$

Taking into account that $\frac{\partial b_2(y)}{\partial \rho} = \nabla b_2(y) \cdot \sigma$ we can write

$$\begin{aligned} u(x) = & \int_{B_e^*(r)} \nabla \cdot (G_e(x, y) k(y) b_2(y) \sigma) dy - \int_{B_e^*(r)} \nabla G_e(x, y) \cdot \sigma k(y) b_2(y) dy \\ & - \int_{B_e^*(r)} G_e(x, y) \nabla k(y) \cdot \sigma b_2(y) dy - \int_{B_e^*(r)} G_e(x, y) k(y) b_2(y) \nabla \cdot \sigma dy. \end{aligned}$$

We may now delete a neighbourhood of x and let it tend to zero to get rid of the singularity of G_e around $x = y$. The first term equals

$$\int_{\partial B_e^*(r)} G_e(x, y) k(y) b_2(y) \sigma \cdot \nu(y) d\sigma(y)$$

which, taking into account that, a priori, $0 \leq b_2 \leq 1$ by the maximum principle, it can be estimated as $O(r)$. Similarly, the second term is $O(r)$, the third is $O(r^2)$ and the last $O(r)$. \square

2.3 Local energy inequality

In this section we present a local energy inequality (c.f. Section 2.1) that will be used to provide the needed oscillation decay in Subsections 2.4.1-2.4.2. Notice that at this stage we know that our weak solution θ is actually in $L_t^\infty L_x^2$ and $L_t^\infty L_x^\infty$, therefore $\theta \in L_t^\infty L_x^{2n}$, which implies that $u = R_g^\perp \theta$ is uniformly bounded in $L_t^\infty L_x^{2n}$.

Remark 2.3. In [CV10a] the authors exploited that the drift $u \in L_t^\infty BMO_x(\mathbb{R}^n)$, preserved under the natural scaling of the equation. Our approach on the other hand is scale dependent and we will use a localized version instead.

It is useful to think of the fractional Laplace-Beltrami as the boundary value of a derivative through a fractional heat equation, namely

$$\begin{cases} \partial_z f^*(x, t, z) = -\Lambda^\alpha f^*(x, t, z), \\ f^*(x, t, 0) = f(x, t), \end{cases}$$

where we denote z the “time” variable since we are dealing already with another time variable t . Notice $\partial_z f^*(x, t, 0) = -\Lambda^\alpha f(x, t)$. An additional feature when $\alpha = 1$ is that

$$(\partial_z^2 + \Delta_g) f^* = 0,$$

which shows harmonicity for f^* , the extension of f . This will be a recurrent theme in the sequel. As a consequence of this observation one may use Green’s identities in the presence of Λ , so long as one is willing to work with f^* instead, allowing the

treatment of this nonlocal operator as a local one (cf. [CS07; CS17]). This idea is exploited deeply in the following lemma:

Lemma 2.3 (Local energy inequality). *Let θ_k satisfy*

$$\partial_t \theta_k + u \cdot \nabla_g \theta_k \leq -\Lambda \theta_k \quad (2.2)$$

and denote $I(z_0) = [0, z_0]$. Let the function $\eta \theta_k^(x, t, z)$ be vanishing in $M \times [0, \infty) \setminus B_g(h) \times I(z_0)$. Then if u satisfies*

$$\sup_{t \in (s, t)} \int_{B_g(h)} |u(x, t)|^{2n} d\text{vol}_g(x) \leq Ch^n, \quad (2.3)$$

and $s \leq t$, the following holds

$$\begin{aligned} & \int_s^t \int_{I(z_0)} \int_{B_g(h)} |\nabla_{x,z}(\eta \theta_k^*)(x, t, z)|^2 d\text{vol}_g(x) dz dt + \int_{B_g(h)} (\eta \theta_k)^2(x, t) d\text{vol}_g(x) \\ & \leq C \left\{ \int_{B_g(h)} (\eta \theta_k)^2(x, s) d\text{vol}_g(x) + h \int_s^t \int_{B_g(h)} |\nabla_x \eta \theta_k|^2 d\text{vol}_g(x) dt \right. \\ & \quad + \int_s^t \int_{I(z_0)} \int_{B_g(h)} |\nabla_{x,z} \eta \theta_k^*|^2 d\text{vol}_g(x) dz dt \\ & \quad \left. + \int_s^t \int_{B_g(h)} (\eta \theta_k)^2(x, t) d\text{vol}_g(x) dt \right\}. \end{aligned}$$

Remark 2.4. Some comments are in order to explain the notation we have adopted to state the lemma. The function θ_k in practice will denote a truncation of the weak solution θ at some level ℓ_k (see Subsection 2.4.1 for the precise definition). Notice that θ_k^* refers to the truncation at the same level of θ^* , the extension, which should not be confused with the extension of the truncation $(\theta_k)^*$ (which will never be used). The gradient $\nabla_{x,z}$ denotes the gradient in the product space $\partial_z + \nabla_g$. We are also making some abuse of notation by denoting with t the time variable and the time integration variable, but we hope no confusion arises.

Proof of Lemma 2.3. As a consequence of subharmonicity of θ_k^* , we get

$$\int_{I(z_0)} \int_{B_g(h)} \eta^2 \theta_k^* (\partial_z^2 + \Delta_g) \theta_k^* d\text{vol}_g(x) dz \geq 0,$$

which yields

$$\begin{aligned} \int_{I(z_0)} \int_{B_g(h)} |\nabla_{x,z}(\eta \theta_k^*)|^2 d\text{vol}_g(x) dz & \leq \int_{I(z_0)} \int_{B_g(h)} |\nabla_{x,z} \eta \theta_k^*|^2 d\text{vol}_g(x) dz \\ & \quad + \int_{B_g(h)} \eta^2 \theta_k \Lambda \theta_k d\text{vol}_g(x). \end{aligned}$$

After integration by parts in the last integral, only one of the appearing boundary integrals does not vanish. It can be majorized, using (2.2), by

$$-\frac{1}{2} \left\{ \frac{\partial}{\partial t} \int_{B_g(h)} \eta^2 \theta_k^2 d\text{vol}_g(x) + \int_{B_g(h)} \nabla_x(\eta^2) \cdot u \theta_k^2 d\text{vol}_g(x) \right\}.$$

Integrating the resulting equality in the time interval $[s, t]$ one gets

$$\begin{aligned} & \int_s^t \int_{I(z_0)} \int_{B_g(h)} |\nabla_{x,z}(\eta \theta_k^*)(x, t, z)|^2 d\text{vol}_g(x) dz dt + \frac{1}{2} \int_{B_g(h)} (\eta \theta_k)^2(x, t) d\text{vol}_g(x) \\ & \leq \int_s^t \int_{I(z_0)} \int_{B_g(h)} |\nabla_{x,z} \eta \theta_k^*|^2 d\text{vol}_g(x) dz dt + \frac{1}{2} \int_{B_g(h)} (\eta \theta_k)^2(x, s) d\text{vol}_g(x) \\ & \quad + \left| \int_s^t \int_{B_g(h)} \nabla_x(\eta^2) \cdot u \theta_k^2 d\text{vol}_g(x) dt \right|. \end{aligned}$$

To arrive to the desire inequality, we just need to deal with the last term. Using Hölder and Cauchy-Schwarz inequalities we get the bounds

$$\begin{aligned} \left| \int_s^t \int_{B_g(h)} \nabla_x(\eta^2) \cdot u \theta_k^2 d\text{vol}_g(x) dt \right| & \leq \int_s^t \|\chi_{B_g(h)} \eta \theta_k\|_{L^{\frac{2n}{n-1}}(M)} \|\nabla_x \eta \cdot u \theta_k\|_{L^{\frac{2n}{n+1}}(M)} dt \\ & \leq \epsilon \int_s^t \|\chi_{B_g(h)} \eta \theta_k\|_{L^{\frac{2n}{n-1}}(M)}^2 dt + \frac{1}{\epsilon} \int_s^t \|\nabla_x \eta \cdot u \theta_k\|_{L^{\frac{2n}{n+1}}(M)}^2 dt, \end{aligned}$$

where $\epsilon > 0$ will be chosen later. Let us treat each term separately. For the first term we use the Sobolev embedding $H^{\frac{1}{2}} \hookrightarrow L^{\frac{2n}{n-1}}$ and the self-adjointness of Λ to obtain

$$\begin{aligned} \int_s^t \|\chi_{B_g(h)} \eta \theta_k\|_{L^{\frac{2n}{n-1}}(M)}^2 dt & \leq \int_s^t \int_M |\Lambda^{\frac{1}{2}}(\chi_{B_g(h)} \eta \theta_k)|^2 d\text{vol}_g(x) dt \\ & \quad + \int_s^t \int_{B_g(h)} (\eta \theta_k)^2(x, t) d\text{vol}_g(x) dt. \end{aligned}$$

The second summand is harmless if $\epsilon \leq C$ while the first is bounded by

$$\int_s^t \int_{B_g(h)} \eta \theta_k \Lambda(\chi_{B_g(h)} \eta \theta_k) d\text{vol}_g(x) dt = - \int_s^t \int_M (\chi_{B_g(h)} \eta \theta_k)^* \partial_z (\chi_{B_g(h)} \eta \theta_k)^* d\text{vol}_g(x) dt \Big|_{z=0}.$$

Now using Green's identities and the decay at infinity, the above integral equals

$$\int_s^t \int_0^\infty \int_M |\nabla_{x,z}(\chi_{B_g(h)} \eta \theta_k)^*|^2 d\text{vol}_g(x) dz dt,$$

and Dirichlet principle implies that it is bounded by

$$\int_s^t \int_0^\infty \int_M |\nabla_{x,z}(\chi_{B_g(h)} \eta \theta_k^*)|^2 d\text{vol}_g(x) dz dt.$$

Indeed, the harmonic extension is a minimizer for the Dirichlet energy functional and this leads immediately to

$$\int_s^t \|\chi_{B_g(h)} \eta \theta_k\|_{L^{\frac{2n}{n-1}}(M)}^2 dt \leq \int_s^t \int_{I(z_0)} \int_{B_g(h)} |\nabla_{x,z}(\eta \theta_k^*)|^2 d\text{vol}_g(x) dz dt,$$

which can be absorbed by the left hand side of the inequality choosing an adequate ϵ . The second term can be handled as follows

$$\begin{aligned} \int_s^t \|\nabla_x \eta \cdot u \theta_k\|_{L^{\frac{2n}{n+1}}(M)}^2 dt &= \int_s^t \left(\int_M |\nabla_x \eta \cdot u \theta_k|^{\frac{2n}{n+1}} d\text{vol}_g(x) \right)^{\frac{n+1}{n}} dt \\ &\leq \int_s^t \left(\int_{B_g(h)} |u|^{2n} d\text{vol}_g(x) \right)^{\frac{1}{n}} \left(\int_{B_g(h)} |\nabla_x \eta \theta_k|^2 d\text{vol}_g(x) \right) dt \\ &\leq \|u\|_{L_t^\infty L_x^{2n}(B_g(h))}^2 \int_s^t \int_{B_g(h)} |\nabla_x \eta \theta_k|^2 d\text{vol}_g(x) dt \end{aligned}$$

where we have used Hölder's inequality and the fact that $\nabla \eta$ is supported in $B_g(h)$. \square

2.4 Hölder regularity

This section will deal with the Hölder continuity of weak solutions. The approach is based on the decrease of the L^∞ norm of either the positive part or the negative part of θ , which implies a decrease in the oscillation. The proof is subdivided in two stages each one containing a step towards the result. In the first we study the decrease under small mean energy hypothesis while in the second we remove such a restriction. This is reminiscent of De Giorgi's work on Hilbert's 19th problem, [DG57].

2.4.1 Small mean energy

To state precisely the concrete part of the proof that we will be dealing with in this subsection, it is convenient to introduce the notation $Q_g(h)$ to denote the pairs (x, t) such that $x \in B_g(h)$ and $t \in t^* + I(h)$, where $t^* \geq t_0$. Following Section 2.2 we use $Q_g^*(h)$ to denote the set of (x, t, z) where $(x, t) \in Q_g(h)$ and $z \in I(h)$. Notice that we are not accurate about the precise position of the time interval, since we only care about its length. In the sequel time intervals will be chosen carefully.

Proposition 2.2. *For h small enough, there exist $\epsilon > 0$ and $\gamma < 1$ both independent of the scale h , so that for any solution θ satisfying*

$$\int_{Q_g(2h)} (\theta)_+^2 d\text{vol}_g(x) dt \leq \epsilon \int_{Q_g(2h)} d\text{vol}_g(x) dt,$$

and

$$\int_{Q_g^*(2h)} (\theta^*)_+^2 d\text{vol}_g(x) dt \leq \epsilon \int_{Q_g^*(2h)} d\text{vol}_g(x) dz dt,$$

we have that

$$\|\theta_+\|_{L^\infty(Q_g(h))} \leq \gamma \|\theta^*\|_{L^\infty(Q_g^*(2h))}.$$

Remark 2.5. Some comments are needed before proceeding to the proof itself. The statement is written in terms of L^∞ bounds because they are related to the oscillation decrease as follows. Indeed, instead of considering θ^* one may look at $\theta^* - a$ for any arbitrary constant a where the resulting function has the same oscillation and satisfies the same drift equation. As a consequence of this one may choose a in such a way that the L^∞ -norm of $(\theta^* - a)_+$ and the oscillation of $\theta^* - a$ are comparable and the decrease on the oscillation is strictly smaller than one. In fact, one may choose a so that $\|(\theta^* - a)_+\|_{L^\infty(Q_g^*(h))}$ equals $\|(\theta^* - a)_-\|_{L^\infty(Q_g^*(h))}$ and hence both are precisely $\frac{1}{2} \text{osc}_{Q_g^*(h)} \theta^*$, where $(f)_- = -(-f)_+$. Hence, provided the conclusion of Proposition 2.2 is true one might bound

$$\begin{aligned} \text{osc}_{Q_g(h/2)} \theta &= \text{osc}_{Q_g(h/2)} (\theta - a) \leq \|(\theta - a)_+\|_{L^\infty(Q_g(h/2))} + \|(\theta - a)_-\|_{L^\infty(Q_g(h/2))} \\ &\leq \gamma \|(\theta^* - a)_+\|_{L^\infty(Q_g(h))} + \|(\theta - a)_-\|_{L^\infty(Q_g(h))} \\ &\leq \frac{1}{2} (1 + \gamma) \text{osc}_{Q_g^*(h)} (\theta^* - a) = \frac{1}{2} (1 + \gamma) \text{osc}_{Q_g^*(h)} \theta^*. \end{aligned}$$

Hence, we have shown as a byproduct that the oscillation would decrease by $(1 + \gamma)/2 < 1$.

We will proceed by a rather tricky and lengthy induction process involving some local energy quantities in the same spirit as in De Giorgi's scheme, but needing to control several boundary terms due to the nonlocality. Since the proof is quite technical and intrincated, let us expose first the general plan of how to achieve the nonlinear inequality for the local energy E_k we are aiming to. In order to clarify the exposition, we state certain claims whose proof will be postponed to the end of this digression.

Fix some $\gamma < 1$ to be specified later. Let us denote by θ_k and θ_k^* the positive part of the truncations of θ and θ^* respectively at the level

$$\ell_k = \|\theta^*\|_{L^\infty(Q_g^*(2h))} \left(1 - (1 - \gamma) \frac{1 + 2^{-k}}{2} \right).$$

Let η_k be a smooth bump function supported in $B_g(h(1 + 2^{-k}))$ identically one in $B_g(h(1 + 2^{-k-1}))$. The main purpose is to find a nonlinear inequality for the local energy

$$\begin{aligned} E_k &= \sup_{t \in t^* + [-h2^{-k-1}, h]} \int_M (\eta_k \theta_k)^2(x, t) d\text{vol}_g(x) \\ &\quad + \int_{t^* + [-h2^{-k-1}, h]} \int_{I(h\delta^k)} \int_M |\nabla_{x,z}(\eta_k \theta_k^*)(x, t, z)|^2 d\text{vol}_g(x) dz dt, \end{aligned}$$

where we assumed $t^* - t_0 \leq h/4$ (cf. [CV10a]). Otherwise one may shrink the intervals in both definitions, using for the lower extreme $t^* - h2^{-k-k_0}$ instead, for some appropriate $k_0 \geq 0$. The constant $\delta < 1$ is a small parameter to be selected later independently of h . It is obvious that the choice of E_k is motivated by the proof of Section 2.1 and the local energy inequality from Section 2.3 (cf. inequality (*) below). Notice that E_k decrease. Recall that we can not afford fixing a reference scale, and use the iterative scaling arguments to deal with other finer scales. Therefore we need to keep track upon the scale during the proof, and this fact will be crucial for our argument to finally work properly.

Taking mean value for $s \in t^* - [h2^{-k-1}, h2^{-k}]$ in the local energy inequality of Lemma 2.3) for $z_0 = h\delta^k$, $\eta = \eta_k$ and any $t \in t^* + [-h2^{-k-1}, h]$ one obtains

$$\begin{aligned}
& \int_{B_g(2h)} (\eta_k \theta_k)^2(x, t) d\text{vol}_g(x) \\
& + \int_{t^* + [-h2^{-k-1}, t]} \int_{I(h\delta^k)} \int_{B_g(2h)} |\nabla_{x,z}(\eta_k \theta_k^*)|^2 d\text{vol}_g(x) dz dt \\
& \leq C \left\{ \frac{2^k}{h} \int_{t^* + [-h2^{-k}, h]} \int_{B_g(2h)} (\eta_k \theta_k)^2(x, s) d\text{vol}_g(x) ds \right. \\
& \quad + h \int_{t^* + [-h2^{-k}, h]} \int_{B_g(2h)} |\nabla_x \eta_k \theta_k|^2 d\text{vol}_g(x) dt \\
& \quad \left. + \int_{t^* + [-h2^{-k}, h]} \int_{I(h\delta^k)} \int_{B_g(2h)} |\nabla_{x,z} \eta_k \theta_k^*|^2 d\text{vol}_g(x) dz dt \right\}.
\end{aligned} \tag{*}$$

The constant C does not depend on the scale h nor on the truncation step k . Taking supremum on t and using the elementary bound $|\nabla \eta_k| \leq C \frac{2^k}{h} \eta_{k-1}$, which can be assumed to be true for a certain general construction of the bump functions, one obtains

$$\begin{aligned}
E_k \leq C \frac{2^{2k}}{h^2} \left\{ h \int_{t^* + [-h2^{-k}, h]} \int_{B_g(2h)} (\eta_{k-1} \theta_k)^2(x, s) d\text{vol}_g(x) ds \right. \\
\left. + \int_{t^* + [-h2^{-k}, h]} \int_{I(h\delta^k)} \int_{B_g(2h)} (\eta_{k-1} \theta_k^*)^2 d\text{vol}_g(x) dz dt \right\}.
\end{aligned}$$

By construction for any x such that $\theta_k(x) > 0$, one has $\theta_{k-1}(x) \geq C(1 - \gamma)2^{-k-2}$. Using

$$\chi_{\{\eta_{k-1} > 0\}} \chi_{\{\theta_k > 0\}} \leq C \frac{2^k}{1 - \gamma} \theta_{k-1} \eta_{k-2},$$

in the above we get the bound

$$\begin{aligned}
E_k \leq C \frac{2^{3k}}{h^2(1 - \gamma)^{2/n}} \left\{ h \int_{t^* + [-h2^{-k}, h]} \int_M (\eta_{k-2} \theta_{k-1})^{2\frac{n+1}{n}} d\text{vol}_g(x) dt \right. \\
\left. + \int_{t^* + [-h2^{-k}, h]} \int_{I(h\delta^k)} \int_M (\eta_{k-2} \theta_{k-1}^*)^{2\frac{n+1}{n}} d\text{vol}_g(x) dz dt \right\}.
\end{aligned}$$

This will be estimated from above in the same nonlinear way as was done in Section 2.1. To do so let us first claim that $\eta_{k-2}\theta_{k-2}^* = 0$ provided $z \in [h\delta^{k-1}, h\delta^{k-2}]$ (cf. Lemma 2.4)

$$\begin{aligned} \int_{I(h\delta^{k-2})} \int_M |\nabla_{x,z}(\eta_{k-2}\theta_{k-2}^*)|^2 d\text{vol}_g(x) dz &= \int_0^\infty \int_M |\nabla_{x,z}(\eta_{k-2}\theta_{k-2}^* \chi_{I(h\delta^{k-2})})|^2 d\text{vol}_g(x) dz \\ &\geq \int_0^\infty \int_M |\nabla_{x,z}(\eta_{k-2}\theta_{k-2}^*)^*|^2 d\text{vol}_g(x) dz \\ &= - \int_M \eta_{k-2}\theta_{k-2}^* \partial_z(\eta_{k-2}\theta_{k-2}^*) d\text{vol}_g(x) \Big|_{z=0} \\ &= \int_M |\Lambda^{\frac{1}{2}}(\eta_{k-2}\theta_{k-2})|^2 d\text{vol}_g(x), \end{aligned}$$

where we have used the aforementioned claim, the decay at infinity, the harmonicity, and Green's identities. This shows that E_{k-2} dominates the norms $L^2 H^{\frac{1}{2}} \hookrightarrow L^2 L^{2n/(n-1)}$ and $L^\infty L^2$, using Hölder's inequality as in Section 2.1 we obtain that

$$\int_{t^*+[-h2^{-k}, h]} \int_M (\eta_{k-2}\theta_{k-2})^{2(n+1)/n} d\text{vol}_g(x) dt \leq E_{k-2}^{1+1/n}.$$

This suggests that an inequality of the type

$$E_k \leq C \frac{2^{3k}}{(1-\gamma)^{2/n} h} E_{k-2}^{1+1/n},$$

might hold true, which is almost the kind of non linear inequality we would like to use.¹ The estimate of the remaining term, $\|\eta_{k-2}\theta_{k-2}^*\|_{L^{2(n+1)/n}(M)}^{2(n+1)/n}$, can be reduced to the above. However, since this is not immediate, let us show first that for any t

$$\theta_{k+1}^*(x, t, z) \leq (\eta_k \theta_k)^*(x, t, z) \text{ for any } (x, z) \in B_g^*(h(1+2^{-k-1}), h\delta^k),$$

holds provided the claim is true. Indeed, by harmonicity one has the bound

$$\theta_k^*(x, t, z) \leq \int_M \eta_k(y) \theta_k(y, t) G(x, y, z) d\text{vol}_g(y) + \|\theta_k^*\|_{L^\infty(Q_g^*(2h))} b_2(x),$$

for any pair $(x, z) \in B_g^*(h(1+2^{-k-1}), h\delta^k)$, which follows using the maximum principle in the cylinder to majorize θ_k^* by $(\eta_k \theta_k)^*$ in the bottom part of the cylinder (i.e. for $x \in B_g(h(1+2^{-k-1}))$ and $z = 0$). On the other hand, the claim allows to disregard the upper part which is bounded by zero. In the rest of the boundary we use the barriers introduced in Section 2.2, which by construction verify (cf. Lemma 2.2)

$$|b_2(x)| \leq C \left\{ \delta^k (1+2^{-k})^{n-2} 2^{k(n-1)} + h(1+2^{-k}) \right\} \leq (1-\gamma) 2^{-k-2},$$

¹Notice the h in the denominator is harmless since we are working with the hypothesis in the mean. This is based on standard dimensional analysis of physical quantities.

provided h and δ are small enough independently of k . From now on we will suppose that h and δ are such that the above holds true. To continue let us observe that

$$\theta^* - \ell_{k+1} = \theta^* - \ell_k - (1 - \gamma)2^{-k-1}\|\theta^*\|_{L^\infty(Q_g^*(2h))},$$

from which one gets the inequality

$$\theta_{k+1}^*(x, t, z) \leq \int_M \eta_k(y) \theta_k(y, t) G(x, y, z) d\text{vol}_g(y) \text{ for } x \in B_g(h(1 + 2^{-k-1})),$$

which yields $\eta_{k+1}\theta_{k+1}^* \leq (\eta_k\theta_k)^*$. Using this fact we get the estimate

$$\begin{aligned} & \int_{t^*+[-h2^{-k}, h]} \int_{I(h\delta^k)} \int_M (\eta_{k-2}\theta_{k-1}^*)^{2\frac{n+1}{n}} d\text{vol}_g(y) dz dt \\ & \leq \int_{t^*+[-h2^{-k}, h]} \int_{I(h\delta^k)} \int_M |(\eta_{k-3}\theta_{k-3})^*|^{2\frac{n+1}{n}} d\text{vol}_g(y) dz dt \\ & \leq \int_{t^*+[-h2^{-k}, h]} \int_{I(h\delta^k)} \int_M (\eta_{k-3}\theta_{k-3})^{2\frac{n+1}{n}} d\text{vol}_g(y) dz dt. \end{aligned}$$

We have applied Jensen's inequality and the identity $\int_M G(x, y, z) d\text{vol}_g(y) = 1$. This is already known to be bounded nicely in terms of E_k as above, provided the claim holds. We conclude the digression by observing that as a consequence of the decreasing character of the energies one would get the following nonlinear inequality

$$E_k \leq C \frac{2^{3k}}{(1 - \gamma)^{2/n} h} E_{k-3}^{1+1/n}, \quad (2.4)$$

The next step is to prove our claim provided some extra hypothesis is fulfilled. This will be helpful to close the induction later on.

Lemma 2.4. *For $k \geq 0$, the following statement holds:*

$$\theta_{k+1}^*(x, t, z) = 0 \text{ for any } (x, z) \in B_g(h(1 + 2^{-k})) \times (h\delta^{k+1}, z_0), \quad (2.5)$$

provided that $\theta_k^(x, t, z_0) = 0$. Moreover, the energy E_k satisfies*

$$E_k h^{-n} \delta^{-2n(k+1)} \leq C^2 (1 - \gamma)^2 2^{-2(k+1)}. \quad (2.6)$$

Proof of Lemma 2.4. We would like to bound $(\eta_k\theta_k)^*$ in the preceding discussion by $C(1 - \gamma)2^{-k-2}$ which, intertwined with the arguments above, will be enough for our purposes. Now if $(x, z) \in B_g(h(1 + 2^{-k})) \times (h\delta^{k+1}, h\delta^k)$ and $t \in t^* + I(h)$ then we can estimate the other term appropriately, indeed:

$$\int_M \eta_k(y) \theta_k(y, t) G(x, y, z) d\text{vol}_g(x) \leq \sqrt{E_k} \|G(x, y, z)\|_{L_x^2(M)}.$$

In the next, we will make use of the following expansion of the fractional heat kernel (cf. [CM15])

$$G(x, y, z) = \sum_{i=0}^{\infty} e^{-\lambda_i z} Y_i(x) Y_i(y),$$

which combined with the local Weyl estimates (cf. Theorem 3.3.1, [Sog14] p. 53)

$$\sum_{\lambda_i \leq \lambda} Y_i(x) Y_i(y) = O(\lambda^n),$$

and summation by parts yields

$$G(x, y, h\delta^{k+1}) \leq C(h\delta^{k+1})^{-n}.$$

From this we have that,

$$\|G(x, y, h\delta^{k+1})\|_{L_x^2(M)}^2 \leq C\delta^{-2n(k+1)}h^{-n}.$$

As a consequence of the hypothesis $(\eta_k \theta_k)^*$ is smaller than $C(1-\gamma)2^{-k-2}$. Altogether, combined with

$$\theta_{k+1}^* \leq \theta_k^* - (1-\gamma)2^{-k-1}\|\theta^*\|_{L^\infty(Q_g^*(2h))},$$

shows that θ_{k+1}^* can not be positive in $B_g(h(1+2^{-k})) \times (h\delta^{k+1}, z_0)$. \square

Remark 2.6. We need to choose z_0 in a specific way, say $z_0 = h/2$, in order to start the inductive procedure (indicated below in the proof of Proposition 2.2). Once we have used this, the parameter z_0 will have the form $\delta^k h$ in the k -th step of the induction process. We believe that Lemma 2.4 is a convenient intermediate step to write the induction neatly.

Next we proceed to the uncover the details of the proof.

Proof of Proposition 2.2. We will show that if ϵ is small enough one can choose a positive $\beta < 1$, independent of h , such that

$$E_k \leq \beta^k h^n \text{ holds for any } k \geq 0.$$

In particular E_k tends to zero, proving the statement. The geometrical decay of this ansatz is very convenient in order to check that the hypothesis (2.6) imposed to E_k of Lemma 2.4 is satisfied (recall that this kind of behaviour is quite plausible for sequences satisfying a non linear inequality (2.4) (cf. Section 2.1)). To that end we choose some $\beta < 1$ satisfying the following smallness condition,

$$\beta^k \leq \delta^{2n(k+1)} C(1-\gamma)^2 4^{-k-1},$$

provided δ and γ are fixed already. Moreover we will also impose

$$C \frac{2^{3k}}{(1-\gamma)^{2/n}} \beta^{\frac{k-3}{n}-4} < 1,$$

which is only useful when $k \geq 4n + 4$.

Next we will prove by induction the following predicate:

$$P(k) : E_k \leq \beta^k h^n \text{ and } \eta_k \theta_k^* = 0 \text{ in the set (2.5) of Lemma 2.4.}$$

Due to the shift in the nonlinear inequality (2.4) and our previous arguments, we already know that if predicates $P(k-3)$, $P(k-2)$, $P(k-1)$ are satisfied then $P(k)$ is also fulfilled, provided $k \geq 4n + 4$. Therefore our work is reduced to check that $P(0)$, $P(1)$, \dots , $P(4n+3)$ are satisfied. The first part of the predicate is quite straightforward. Indeed, using the local energy inequality derived in Section 2.3, we can take ϵ verifying

$$\epsilon < C\beta^{4n+3},$$

and hence $E_k \leq \beta^{4n+3} h^n$, for $0 \leq k \leq 4n+3$, as required. Next, let us realize that if we prove the second part of the statement for $k=0$, we would be done. Indeed, appealing to Lemma 2.4, we would prove $P(1)$. Afterwards using $P(1)$ and smallness on E_1 , we deduce $P(2)$. Similarly one gets $P(3)$. Therefore the induction process works nicely without any further assumptions using the non linear inequality (2.4) provided β is smaller than a threshold quantity, which is independent of the scale h as we specified previously.

Let us show how to start the induction at $k=0$. The maximum principle allows us to bound θ^* in $B_g(h) \times I(h)$ as follows

$$\theta^*(x, t, z) \leq \int_M \theta(y, t) \chi_{B_g(h)}(y) G(x, y, z) d\text{vol}_g(y) + \|\theta^*\|_{L^\infty(Q_g^*(2h))} \left(b_1(x, z) + \frac{z}{h} \right) \quad (2.7)$$

where the barrier b_1 has been constructed in Section 2.2. The only problematic term is the first one since the second term can be handled using Lemma 2.1 and the third term can be bounded easily due to its linearity. By Hölder's inequality we get

$$\begin{aligned} \int_M \theta(y, t) \chi_{B_g(h)}(y) G(x, y, z) d\text{vol}_g(y) &\leq C \|\theta^*\|_{L^\infty(Q_g^*(2h))} \|G(x, y, z)\|_2 \|\chi_{B_g(h)}\|_2 \\ &\leq C 2^n. \end{aligned}$$

Using Weyl's law asymptotics, positivity of the characteristic function and fractional heat kernel estimates one gets

$$|\theta^*(x, t, z)| \leq \gamma^* \|\theta^*\|_{L^\infty(Q_g^*(2h))}$$

where $x \in B_g(h)$, $z \in I'(h)$ and $\gamma^* < 1$. This shows that the hypothesis of Lemma 2.4 are satisfied for θ_0^* with $\gamma = \frac{1}{3}(4\gamma^* - 1)$ and $z_0 = \frac{h'}{2}$, where we can take $z_0 \in I'(h) = [\frac{h}{3}, \frac{2h}{3}]$ (see Figure 2.3). \square

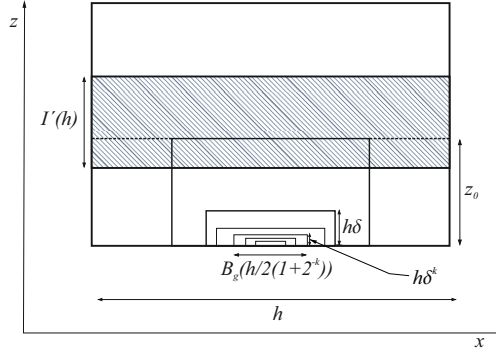


Fig. 2.3.: Initial step and iterative procedure.

Remark 2.7. In the estimate above we are assuming that $C(M)2^n$ is small enough. But if that is not the case, one may use instead of balls decreasing by half, balls decreasing by some fixed small quantity c . Then, the estimate above will have the form $C(M)2^n c^{n/2}$ and we can choose c so that the above estimate is indeed strictly smaller than one.

2.4.2 Arbitrary energy

The purpose of this section is to free Proposition 2.2 from its small mean energy requirement. To do so we prove a version of De Giorgi's isoperimetric inequality following closely the argument in [CV10a] though it needs careful adaptation to avoid problems with the different scales. Let us first introduce some convenient notation: denote by $Q_g^*(h)$ the cube of all $(x, t, z) \in B_g(h) \times I(h) \times I(h)$, $Q_g(h)$ the set $(x, t) \in B_g(h) \times I(h)$ and $|A|$ the measure of A in the product. It would be useful while reading this section to keep in mind that $B_g(h)$, $B_g^*(h)$, $Q_g^*(h)$ are approximately of order h^n , h^{n+1} , h^{n+2} , respectively.

De Giorgi's isoperimetric inequality, in our setting, will relate the measures of the following sets

$$\begin{cases} \mathcal{A}(t) = \{(x, z) \in B_g^*(h) : \theta^*(x, t, z) \leq 0\} \\ \mathcal{B}(t) = \{(x, z) \in B_g^*(h) : \theta^*(x, t, z) \geq \frac{1}{2}\|\theta^*\|_{L^\infty(Q_g^*(h))}\} \\ \mathcal{C}(t) = \{(x, z) \in B_g^*(h) : 0 < \theta^*(x, t, z) < \frac{1}{2}\|\theta^*\|_{L^\infty(Q_g^*(h))}\} \end{cases}$$

It reads as follows:

Lemma 2.5 (De Giorgi's isoperimetric inequality). *For h small enough, the following inequality holds for any function $\theta^* \in H^1(B_g(h))$*

$$|\mathcal{A}(t)||\mathcal{B}(t)| \leq C|\mathcal{C}(t)|^{\frac{1}{2}} K^{\frac{1}{2}} h^{n+2}, \quad (2.8)$$

where $K = \|\nabla_{x,z}\theta^*\|_{L^2(B_g^*(h))}^2$.

Proof of Lemma 2.5. The proof is essentially the same as in in [CV10a; CV10b], but being careful to keep the small scale dependence. \square

Now we can state the main result:

Lemma 2.6. *For any small enough² $\epsilon > 0$ independent of h , there exists $\delta = \delta(\epsilon) > 0$ such that for any weak solution θ satisfying*

$$|\{(x, t, z) \in Q_g^*(2h) : \theta^*(x, z, t) \leq 0\}| \geq \frac{1}{2}|Q_g^*(2h)|.$$

We have that the hypothesis

$$\left| \left\{ (x, t, z) \in Q_g^*(2h) : 0 < \theta^* < \frac{1}{2}\|\theta^*\|_{L^\infty(Q_g^*(h))} \right\} \right| \leq \delta|Q_g^*(2h)|,$$

implies

$$\int_{Q_g(h)} \left(\theta - \frac{1}{2}\|\theta^*\|_{L^\infty} \right)_+^2 d\text{vol}_g(x) dt \leq \epsilon|Q_g(h)|,$$

and

$$\int_{Q_g^*(h)} \left(\theta^* - \frac{1}{2}\|\theta^*\|_{L^\infty} \right)_+^2 d\text{vol}_g(x) dt \leq \epsilon|Q_g^*(h)|.$$

Before proceeding to the proof itself let us glimpse the rough idea behind it. Let us suppose that both $\mathcal{B}(t)$ and $\mathcal{C}(t)$ are smaller than γh^{n+1}

$$\int_{B_g^*(h)} (\theta^*)_+^2(t) dx dz \leq \|\theta^*\|_{L^\infty(B_g^*(h))}^2 (|\mathcal{B}(t)| + |\mathcal{C}(t)|) \leq \gamma C h^{n+1}.$$

Integrating the identity

$$\int_{B_g(h)} \theta_+^2 d\text{vol}_g(x) = \int_{B_g(h)} (\theta^*)_+^2 d\text{vol}_g(x) - 2 \int_0^z \int_{B_g(h)} \theta_+^* \partial_z \theta^* d\text{vol}_g(x) d\bar{z}, \quad (2.9)$$

for $z \in I(h)$ one gets

$$h \int_{B_g(h)} \theta_+^2(x, t) dx \leq \int_{B_g^*(h)} \theta_+^*(x, t, z)^2 dx dz + \int_0^h \int_0^z \int_{B_g^*(h)} \theta_+^*(t) \partial_z \theta^* dx d\bar{z} dz.$$

The first term can be bounded by $O(\gamma h^{n+1})$, using the previous inequality; the second, applying Fubini and Cauchy-Schwarz is bounded by

$$h \|\theta_+^*(t)\|_{L^2(B_g^*(h))} \|\partial_z \theta^*\|_{L^2(B_g^*(h))} \leq \gamma h^{n+1}.$$

²Depending on M, θ_0, t_0 .

Notice that the L^2 -gradient norm might be expected to be of size $h^{(n-1)/2}$, from dimensional considerations. Summarizing: if the above argument works we may integrate the resulting inequality for all times $t \in I(h)$ achieving

$$\int_{Q_g^*(h)} \theta_+^2(x, t) d\text{vol}_g(x) dt = O(\sqrt{\gamma} h^{n+2}).$$

Notice that this estimate is stronger than the one we intend to prove. In fact, the assumption above should not be expected to hold for any $t \in I(h)$, the proof will show that an elaboration of the aforementioned argument reaching control on the size of the corresponding sets actually holds for most of the times $t \in I(h)$. The remaining times, for which it does not hold, will have a controlled size.

Proof of Lemma 2.6. From De Giorgi's isoperimetric inequality (2.8) one notice that one may control $\mathcal{B}(t)$'s smallness if one knows for an appropriate K that $\mathcal{A}(t)$ is big, while $\mathcal{C}(t)$ is small by hypothesis. Let us introduce a subset of times for which we do expect some control of \mathcal{B} due to De Giorgi's isoperimetric inequality provided we manage to prove $\mathcal{A}(t)$ is big enough:

$$\mathcal{T} = \left\{ t \in I(h) : \int_{B_g^*(h)} |\nabla \theta_+^*|^2 d\text{vol}_g(x) dz \leq K \text{ and } |\mathcal{C}(t)|^{\frac{1}{2}} \leq 2\epsilon^3 h^{\frac{n+1}{2}} \right\}.$$

The complement of this set is small in $I(h)$, in the sense that it is smaller than $\epsilon h/2$, choosing

$$K = \frac{4}{\epsilon h} \int_{Q_g^*(h)} |\nabla_{x,z} \theta_+^*|^2 d\text{vol}_g(x) dz dt.$$

Indeed, define $\delta = \epsilon^8$, one may obtain the following weak bound

$$\left| \left\{ t \in I(h) : |\mathcal{C}(t)|^{\frac{1}{2}} \geq 2\epsilon^3 h^{\frac{n+1}{2}} \right\} \right| \leq \frac{1}{4\epsilon^6 h^{n+1}} \int_{I(h)} |\mathcal{C}(t)| dt \leq \epsilon^2 h/4.$$

Furthermore, the control on the remaining condition is provided by the following weak bound

$$\left| \left\{ t \in I(h) : \int_{B_g^*(h)} |\nabla_{x,z} \theta_+^*|^2 d\text{vol}_g(x) dz \geq K \right\} \right| \leq \epsilon h/4.$$

Along the proof several smallness assumptions will be imposed on ϵ , being a finite number this causes no problem for the argument to work. Notice that if $t \in I(h) \cap \mathcal{T}$ is such that $|\mathcal{A}(t)| \geq \frac{1}{4} |B_h^*|$ then using De Giorgi's as above we obtain

$$\int_{I(h)} \int_{B_g^*(h)} |\nabla_{x,z} \theta_+^*|^2 d\text{vol}_g(x) dz dt \leq C h^n$$

which follows from the local energy inequality, cf. Section 2.3. Therefore one gets

$$|\mathcal{B}(t)| \leq \epsilon^{\frac{5}{2}} C h^{n+1},$$

by De Giorgi's isoperimetric inequality. This leads to $\|\theta\|_{L^2(Q_g(h))} \leq \epsilon h^{\frac{n+1}{2}}$ and $\|\theta^*\|_{L^2(Q_g^*(h))} \leq \epsilon h^{\frac{n}{2}+1}$. We claim that such estimate is true for some $t_1 \in \mathcal{T}$ which does not lie on

$$\left[-\frac{h}{4} + t^*, t^*\right] \subseteq [t^* - h, t^*] = I(h).$$

Indeed, the non existence of t^* leads to a contradiction with the size condition in the statement. We will use this facts to prove the following:

Claim 2.1. *The inequality $|\mathcal{A}(t)| \geq \frac{\epsilon^2}{2} |B_g^*(h)|$ holds for any $t \in I(h/4) \cap \mathcal{T}$.*

The local energy inequality assures that for $t \geq t_1$ one has

$$\int_{B_g(h)} \theta_+^2(t) d\text{vol}_g(x) \leq \int_{B_g(h)} \theta_+^2(t_1) d\text{vol}_g(x) + C \|\theta\|_{L^\infty}^2 (t - t_1) (h^{n-1} + h^n).$$

Observe that $t - t_1$ is of order h ; let $t - t_1 \leq \epsilon h$, if ϵ is small enough

$$\|\theta_+\|_{L^2(B_g(h))}^2 \leq \frac{1}{100} h^n.$$

Now, for such t

$$\begin{aligned} \theta_+^*(x, t, z) &= \theta_+(x, t) + \int_0^z \partial_z \theta_+^* d\bar{z} \\ &\leq |\theta_+(x, 0)| + \sqrt{z} \left(\int_0^z |\partial_z \theta_+^*|^2 d\bar{z} \right)^{\frac{1}{2}}. \end{aligned}$$

Applying the above with $z \leq \epsilon^2 h$ implies

$$\|\theta_+^*(x, t, z)\|_{L^2(B_g(h))} \leq \|\theta_+(x, t)\|_{L^2(B_g(h))} + \epsilon h^{\frac{1}{2}} \left(\int_{B_g^*(h)} |\nabla_z \theta^*(x, t, \bar{z})|^2 d\bar{z} d\text{vol}_g(x) \right)^{\frac{1}{2}}.$$

Our previous results show that it is bounded by $\frac{1}{100} h^n + \epsilon C h^{\frac{n}{2}}$ which is smaller than $\frac{1}{2} |B_g(h)|^{\frac{1}{2}}$ if ϵ is small enough. Application of a weak L^2 inequality implies

$$\left| \left\{ x \in B_g(h) : \theta_+^*(x, t, z) \geq \frac{1}{2} \|\theta^*\|_{L^\infty(B_g^*(h))} \right\} \right| \leq \frac{1}{4} |B_g(h)|.$$

Integrating $z \in I(\epsilon^2 h)$ yields

$$\left| \left\{ x \in B_g(h), z \in I(\epsilon^2 h) : \theta_+^*(x, t, z) \geq \frac{1}{2} \|\theta^*\|_{L^\infty(B_g^*(h))} \right\} \right| \leq \frac{\epsilon^2}{4} |B_g^*(h)|$$

Since $t \in \mathcal{T}$ one has that $|\mathcal{C}(t)| \leq 2\epsilon^6 h^{n+1} \leq \epsilon^5 |B_g^*(h)|$ and we get the following estimate from below

$$\begin{aligned} |\mathcal{A}(t)| &\geq |B_g(h) \times I(\epsilon^2 h)| - |\mathcal{C}(t)| \\ &\quad - \left| \left\{ x \in B_g(h), z \in I(\epsilon^2 h) : \theta_+^*(x, t, z) \geq \frac{1}{2} \|\theta^*\|_{L^\infty(B_g^*(h))} \right\} \right| \\ &\geq \left(\epsilon^2 \left(1 - \frac{1}{4} \right) - \epsilon^5 \right) |B_g^*(h)| \geq \frac{1}{2} \epsilon^2 |B_g^*(h)|, \end{aligned}$$

proving our claim, provided that ϵ is small enough.

Repeating the argument at the beginning of the proof one gets $|\mathcal{B}(t)| \leq C\sqrt{\epsilon}h^{n+1}$. So, finally, we have proved that, in fact, $|\mathcal{A}(t)| \geq \frac{1}{4}|B_g^*(h)|$ for any $t \in \mathcal{T}$ satisfying $t - t_1 \leq \epsilon h$. Then, as before:

$$\begin{aligned} |\mathcal{A}(t)| &\geq |B_g^*(h)| - |\mathcal{B}(t)| - |\mathcal{C}(t)| \\ &\geq \left(1 - C\sqrt{\epsilon} - \epsilon^5 \right) |B_g^*(h)| \geq \frac{1}{4} |B_g^*(h)|, \end{aligned}$$

which holds provided ϵ is small enough. This shows that $|\mathcal{A}(t)| \geq \frac{1}{4}|B_g^*(h)|$ holds in $t_1 + I(\epsilon h) \cap \mathcal{T}$. We may change t_1 to some t_2 to its left provided \mathcal{T}^c is small compared to ϵh , as it actually happens. Proceeding in this way one covers $\mathcal{T} \cap I(h/4)$. As a consequence

$$|\mathcal{B}(t)| \leq \epsilon^2 |B_g^*(h)|,$$

holds for any $t \in \mathcal{T} \cap I(h/4)$. And this allows us to estimate

$$\begin{aligned} \int_{Q_g^*(h)} \left(\theta^* - \frac{1}{2} \|\theta^*\|_{L^\infty} \right)_+^2 d\text{vol}_g(x) dz dt &= \|\theta^*\|_{L^\infty}^2 \left(\frac{1}{2} \int_{t \in \mathcal{T}} |\mathcal{B}(t)| dt + \int_{t \notin \mathcal{T}} 1 dt \right) \\ &\leq C(\epsilon^2 + \epsilon) |Q_g^*(h)|. \end{aligned}$$

To get the other smallness condition one uses equation (2.9), with $(\theta^* - 1)_+$ instead, so that

$$\begin{aligned} \int_{Q_g(h)} \left(\theta - \frac{1}{2} \|\theta^*\|_{L^\infty} \right)_+^2 dx dt &= \int_{Q_g(h)} \left(\theta^* - \frac{1}{2} \|\theta^*\|_{L^\infty} \right)_+^2 d\text{vol}_g(x) dt \\ &\quad - 2 \int_0^h \int_0^z \int_{B_g(h)} \left(\theta^* - \frac{1}{2} \|\theta^*\|_{L^\infty} \right)_+ \partial_z \left(\theta^* - \frac{1}{2} \|\theta^*\|_{L^\infty} \right)_+ d\text{vol}_g(x) d\bar{z} dt. \end{aligned}$$

However from the previous argument we already know that there exist some $z \in I(h)$ such that the first integral is $\sqrt{\epsilon}h^{n+1}$. By Cauchy-Schwarz inequality the other term is dominated by

$$\left\| \left(\theta^* - \frac{1}{2} \|\theta^*\|_{L^\infty} \right)_+ \right\|_{L^2(Q_g^*(h))} \left\| \partial_z \left(\theta^* - \frac{1}{2} \|\theta^*\|_{L^\infty} \right)_+ \right\|_{L^2(Q_g^*(h))}.$$

Finally, the last term can be bounded by the L^2 norm of the gradient $\nabla_{x,z}\theta^*$ which is controlled by $h^{\frac{n-1}{2}}$. All this altogether proves

$$\left\| \left(\theta - \frac{1}{2} \|\theta^*\|_{L^\infty} \right) \right\|_{L^2(Q_g(h))} \leq \sqrt{\epsilon} h^{\frac{n+1}{2}},$$

which making ϵ even smaller, if necessary, implies Lemma 2.6. \square

Let us now explore the consequences of this rather technical lemma:

Proposition 2.3. *For h small enough, there exist a $\gamma < 1$ (independent of the scale h) such that the following holds*

$$\|\theta_+\|_{L^\infty(Q_g(h))} \leq \gamma \|\theta^*\|_{L^\infty(Q_g^*(2h))}.$$

Proof of Proposition 2.3. Without loss of generality we will assume that θ^* is such that $C_0 = -\inf_{Q_g^*(h)} \theta^* = \sup_{Q_g^*(h)} \theta^*$ and such that θ is negative at least for half the points in $Q_g^*(h)$, otherwise one may subtract an appropriate quantity or argue with $-\theta$ instead. We define the following truncations

$$\tau_k = 2^k \left(\theta - \frac{1}{2} (1 - 2^{-k}) C_0 \right)_+.$$

Notice that the extension τ_k^* is precisely $2^k (\theta^* - \frac{1}{2} (1 - 2^{-k}) C_0)_+$ and that all of them are, by construction, bounded by the same C_0 . We claim now that τ_{k_0} satisfies the hypothesis of Lemma 2.6 for some integer $k_0 \leq \frac{1}{\delta(\epsilon)}$. Indeed, otherwise

$$\{\tau_k^* < 0\} = \left\{ \tau_{k-1}^* < \frac{1}{2} C_0 \right\} \geq \{\tau_{k-1}^* < 0\} + \delta |Q_g(h)|,$$

which can not hold inductively longer than $\frac{1}{\delta}$ times. As a consequence, Lemma 2.6 implies that τ_{k_0} is under the hypothesis of Proposition 2.2. Hence

$$\|(\tau_{k_0})_+\|_{L^\infty(Q_g(h/2))} \leq \gamma \|\tau_{k_0}^*\|_{L^\infty(Q_g^*(h))},$$

for some $\gamma < 1$. Unravelling notation one observes that

$$\|\theta_+\|_{L^\infty(Q_g(h))} \leq \left(1 - \frac{1-\gamma}{2^{k_0+1}} \right) \|\theta^*\|_{L^\infty(Q_g^*(2h))}.$$

Notice now that the decrease is smaller than one and that it can be taken independently of the scale h we are working with. \square

2.4.3 Proof of Theorem 1.2

Finally, to conclude the proof of Theorem 1.2, we just need to use again the barrier b_2 plus the linear function, i.e.,

$$\|\theta^*\|_{L^\infty(Q_g^*(h))} b_2 + L(z),$$

where $L(z)$ is linear function interpolating between $L(0) = \|\theta\|_{L^\infty(Q_g(h/2))}$ and $L(h/2) = \|\theta^*\|_{L^\infty(Q_g^*(h))}$ (see Figure 2.4). As a consequence (cf. Lemma 2.2) one show the existence of $\gamma < 1$ such that

$$\|\theta^*\|_{L^\infty(Q_g^*(ch))} \leq \gamma \|\theta^*\|_{L^\infty(Q_g^*(h))},$$

for some sufficiently small positive constant c . This implies the statement with $\alpha = \alpha(c, \|\theta_0\|_{L^2}, M, t_0) > 0$.

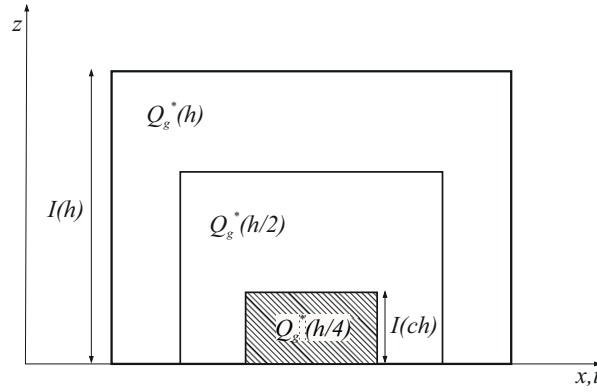


Fig. 2.4.: Scheme of the final interior oscillation decay

2.5 Proof of Theorem 1.1

The main purpose of this section is to modify the arguments used before to prove Theorem 1.2, to deal with the $(SQG)_M$, where $u \notin L_x^\infty$. We provide a geometrical twist of the argument from [CV10a] where the authors use translations and dilations to construct an iterative sequence of auxiliary functions related to θ for which the oscillation decays fast enough to obtain C^α regularity. Until now, the proof worked properly for a general orientable compact manifold. However, to address the problem that concerns us, we need to restrict the ambient space to the n -dimensional sphere \mathbb{S}^n (actually to \mathbb{S}^2). The main reason, is that we use the rich group of symmetries enjoyed by the sphere: the rotations.

Let us begin illustrating the geometrical idea we take advantage of by presenting a glimpse of the method. First we define a new function

$$F(s, x) = \theta(R_s(x), s + t_0),$$

where R_s is a rigid rotation of the sphere around some axis and s denotes the arc length of the particles moving in the corresponding equator. Notice that such an F satisfies the equation

$$\partial_s F + v \cdot \nabla_g F = -\Lambda_g F, \quad (2.10)$$

where the new drift given by $v = u - \dot{R}_s$ and \dot{R}_s is the infinitesimal generator of the rotation R_s . Notice that rotations are global isometries respecting the nonlocal diffusive operator. The upshot is that the new drift v associated to equation (2.10) satisfies the hypothesis (2.3) of Lemma 2.3.

To that purpose, let fix $x_0 \in \mathbb{S}^2$ and a ball $B_g(h)$ around x_0 . We use the standard embedding $\mathbb{S}^2 \subseteq \mathbb{R}^3$. Let $u(x) = u^t(x) + u^n(x)$ where u^t denotes the projection to the tangent plane at x_0 , $T_{x_0}\mathbb{S}^2$. Now, near x_0 it is evident that u^n is small, more precisely

$$u^n(x) = O(ru) \text{ for } x \in B_g(r), \quad (2.11)$$

provided r is small enough. We define R_s as the rotation generated by the following tangent vector at x_0

$$\dot{R}_s(x_0) = \frac{1}{|B_g(h)|} \int_{B_g(h)} u^t(s + t, R_s(x)) d\text{vol}_g(x) \in T_{x_0}\mathbb{S}^2.$$

This definition is equivalent to an ordinary differential equation for R_s with $R_0 = \text{id}$. Notice that, for the same estimate as (2.11)

$$\dot{R}(x) = \dot{R}_s(x_0) + O(r\dot{R}_s(x_0)), \quad (2.12)$$

for any $x \in B_g(r)$ with small r . Hence

$$\dot{R}_s(x_0) = \frac{1}{|B_g(h)|} \int_{B_g(h)} u d\text{vol}_g(x) - \frac{1}{|B_g(h)|} \int_{B_g(h)} u_n d\text{vol}_g(x) = u_{B_g(h)} + O(1), \quad (2.13)$$

where

$$u_{B_g(h)} := \frac{1}{|B_g(h)|} \int_{B_g(h)} u d\text{vol}_g(x).$$

Indeed, the second integral term in (2.13) can be estimated as

$$\begin{aligned} \frac{1}{|B_g(h)|} \int_{B_g(h)} u_n d\text{vol}_g(x) &\leq \frac{1}{|B_g(h)|^{1/2}} \left(\int_{B_g(h)} |u_n|^2 d\text{vol}_g(x) \right)^{1/2} \\ &\leq \left(\int_{B_g(h)} |u|^2 d\text{vol}_g(x) \right)^{1/2} = O(1) \end{aligned} \quad (2.14)$$

where we used Cauchy-Schwarz inequality, the extra h from the estimate above (2.12) and the L^2 boundedness of the Riesz transform. This is where the two dimensionality of the sphere becomes crucial in the argument. For higher dimensional spheres the bound is not good enough. This can be seen very easily since the n -dimensional analogue of estimate (2.14) is given by

$$h^{1-\frac{n}{2}} \left(\int_{B_g(h)} |u|^2 d\text{vol}_g(x) \right)^{1/2},$$

which becomes only an $O(1)$ if $n = 2$. We can now check the hypothesis (2.3) of Lemma 2.3, given by

$$\sup_{t \in (s, t)} \frac{1}{|B_g(h)|} \int_{B_g(h)} |v|^4 d\text{vol}_g(x) \leq Ch^2.$$

Therefore,

$$\begin{aligned} \left(\frac{1}{|B_g(h)|} \int_{B_g(h)} |u - \dot{R}_s|^4 d\text{vol}_g(x) \right)^{1/4} &\leq \left(\frac{1}{|B_g(h)|} \int_{B_g(h)} |u - u_{B_g(h)}|^4 d\text{vol}_g(x) \right)^{1/4} \\ &\quad + \left(\frac{1}{|B_g(h)|} \int_{B_g(h)} |u_{B_g(h)} - \dot{R}_s(x_0)|^4 d\text{vol}_g(x) \right)^{1/4} \\ &\quad + \left(\frac{1}{|B_g(h)|} \int_{B_g(h)} |\dot{R}_s(x_0) - \dot{R}_s|^4 d\text{vol}_g(x) \right)^{1/4} \\ &= \mathcal{O}(1) \end{aligned}$$

where the first term is bounded due to the John-Nirenberg inequality since $u \in BMO(\mathbb{S}^2, \mathbb{R}^3)$ (cf. [BN95; Tay09]). The second is $O(1)$ follows from the relation (2.13), while the third is bounded by $h|\dot{R}_s(x_0)|$, which is again $O(1)$ by Cauchy-Schwarz inequality due to the extra h as the estimate (2.14), proving the assertion. Now it is licit to infer from Theorem 1.2 the existence of some $\gamma < 1$ for which

$$\text{osc}_{Q_g(ch)} F(s, x) \leq \gamma \cdot \text{osc}_{Q_g(2h)} F(s, x).$$

Notice that this can not be rephrased directly in terms of the oscillation of $\theta(x, t)$, since to do so, we need to control the displacements produced by the rotations $R_s(x)$. One might, nevertheless, bound it from below paying the price of making times smaller (not of order h , but some power fraction $h^{\frac{1}{m}}$) to compensate this displacements. The a priori bound for the displacement is given by

$$\begin{aligned} R_s(x) &\leq s \sup_{s \leq h} \dot{R}_s(x) \leq \frac{s}{h^2} \sup_{s \leq h} \int_{B_g(h)} u d\text{vol}_g(x) \\ &\leq Ch^{2/p'-1} \|u\|_{L^p(M)}, \end{aligned}$$

where p' is the conjugate of p . Therefore, choosing $p > 2$ we can estimate it from above by, say, $h^{1/3}$. As a consequence, the decrease in the oscillation for $\theta(x, t)$ is of the form

$$\text{osc}_{Q_g(h^K)}\theta(x, t) \leq \gamma \cdot \text{osc}_{Q_g(h)}\theta(x, t),$$

for some K big enough and h small enough (depending on some fixed quantities). This implies a modulus of continuity of the form $\omega(\rho) = \log(1/\rho)^{-\alpha}$ with ρ the geodesic distance $d_g(x, y)$ for some $\alpha = \alpha(t_0, \|\theta_0\|_{L^2(\mathbb{S}^2)})$ which deteriorates as t_0 approaches zero. This concludes the proof of Theorem 1.1.

Global well-posedness of the critical SQG on the sphere

The third chapter deals with the global-well posedness in Sobolev spaces of the critical SQG equation on the two dimensional sphere, stated in Theorem 1.3. To do so, we will prove first some nonlinear maximum principles for the fractional laplacian on the sphere which combined with the modulus of continuity proven in Theorem 1.1, will be enough to close the arguments, yielding Theorem 1.3.

3.1 Integral representation, pointwise estimates and stereographic projection

In this section we provide several observations, technical lemmas and tools that will be instrumental in the sequel. First of all, let us present one of the main devices of the proof that is of independent interest. As usual let (M, g) be a compact manifold of dimension $n \geq 2$ whose Laplace-Beltrami operator is denoted by $-\Delta_g$. Then we have the following integral representation

Theorem 3.1. *Let f be smooth and $\alpha \in (0, 1)$, then for a sufficiently large parameter N one has the following representation*

$$(-\Delta_g)^\alpha f(x) = P.V. \int_M \frac{f(x) - f(y)}{d_g(x, y)^{n+2\alpha}} (c_{n,s} \chi u_0 + k_N)(x, y) d\text{vol}_g(y) + \mathcal{O}(\|f\|_{H^{-N}(M)}), \quad (3.1)$$

where $k_N(x, y) = \mathcal{O}(d_g(x, y))$ is a smooth function, χ is a smooth cut off function equal to one the diagonal and supported around it. The implicit constant depends on N , $c_{\alpha,n} > 0$ is a constant independent of N and $u_0(x, x) = 1$.

Notice that the norm in the error might be taken to be in L^∞ . The proof relies on spectral calculus intertwined with Hadamard parametrix plus a harmless error, [AO+18c]. Let us remark that the explicit smoothing effect in the error term is crucial when using this representation to prove the global well-posedness of strong solution for the critical SQG, as we will see. One may compare this expression with the well-known ones in the case of torus or Euclidean space [CC04; CC03].

Since, we will use sometimes the integral representation for negative index α , let us also write the explicit representation:

Theorem 3.2. For $\alpha \in (-1, 0)$ and under the hypothesis of Theorem 3.1,

$$(-\Delta_g)^\alpha f(x) = P.V. \int_M \frac{f(y)}{d_g(x, y)^{n+2\alpha}} (c_{n,\alpha} \chi_{u_0} + k_N)(x, y) d\text{vol}_g(y) + \mathcal{O}(\|f\|_{H^{-N-2}(M)}). \quad (3.2)$$

Next, we need to prove a pointwise commutator estimate related to the ones appearing in the work of Constantin and Ignatova (cf. [CI16; CI17]) which allow us to estimate $[\Lambda_g, \nabla_g]$. This commutator involves the action of Λ_g , a pseudodifferential operator on fiber bundles, which can be defined in several ways. However, our pointwise estimate do not seem to be an immediate consequence of this general setting.

Lemma 3.1. Let f be a smooth function on (M, g) , $\alpha \in (0, 2)$ and suppose that the smooth function a satisfies $a(x) - a(y) = O(d_g(x, y)^2)$. Then there is a constant $C = C(n, \alpha)$ such that the following pointwise commutator estimate holds

$$|[\Lambda_g^\alpha, a]f(x)| \leq C\|f\|_\infty. \quad (3.3)$$

Proof of Lemma 3.1. Let us employ the following representation of the fractional operator (up to a constant)

$$\Lambda_g^\alpha f(x) = \int_0^\infty t^{-1-\alpha/2} \left(f(x) - \int_M G(x, y, t) f(y) d\text{vol}_g(y) \right) dt,$$

from which it easily follows that the commutator satisfies

$$[\Lambda_g^\alpha, a]f(x) = \int_0^\infty t^{-1-\alpha/2} \int_M G(x, y, t) (a(y) - a(x)) f(y) d\text{vol}_g(y) dt.$$

Which might be estimated for small times quite crudely employing the following upper bound for the heat kernel (cf. [LY86], Corollary 3.1)

$$G(x, y, t) \leq C(M, g) \frac{e^{-\frac{d_g(x, y)^2}{5t}}}{t^{n/2}}.$$

The proof concludes observing that one can also estimate the rest easily taking advantage of the exponential decay of the heat kernel on compact manifolds for large times. \square

Remark 3.1. Alternatively one may actually provide a proof just computing the commutator using our kernel representation (cf. Theorem 3.1.)

Before proceeding any further let us include the following general result:

Lemma 3.2. Let $f \geq 0$ be some smooth function on (M, g) and denote by $\bar{x} \in M$ the point where it reaches its maximum. Then, for any $\alpha \in (0, 2)$

$$\Lambda_g^\alpha f(\bar{x}) \geq 0.$$

Remark 3.2. *This is somehow suprising since curvature might have some effects in view of the representation formula (3.1). It is nevertheless true in the stated generality as we will prove now (cf. [CM15]).*

Proof of Lemma 3.2. Let us introduce the following Cauchy problem for a fractional heat equation on the manifold, namely

$$\begin{cases} \frac{d}{dt}h = -\Lambda_g^\alpha h, \\ h(\cdot, 0) = f. \end{cases}$$

It is well known that it satisfies the following maximum principle

$$\|h(\cdot, t)\|_{L_x^\infty(M)} \leq \|h(\cdot, 0)\|_{L_x^\infty(M)}.$$

As a consequence at that maximum point \bar{x} we have $h(\bar{x}, t) - h(\bar{x}, 0) \leq 0$, dividing then by t and letting t approach zero one gets $\frac{d}{dt}h(\bar{x}, 0) \leq 0$, which is equivalent to our claim. \square

We include now an approximation of which we will take advantage of in the next sections and although it holds in general dimension, for the sake of simplicity, we shall present the details of the proof only in dimension two. We will show that for any point $x \in \mathbb{S}^2$ we can approximate to second order the infinitesimal rotations \dot{R}_1, \dot{R}_2 corresponding to the rotations induced by a given orthonormal system of vectors in $T_x\mathbb{S}^2$ with vector fields ∂_1 and ∂_2 in some appropriate coordinates. Let us consider the stereographic projection with p , its south pole, the origin of coordinates $(0, 0, 0) \in \mathbb{R}^3$. Then, in Cartesian coordinates (x, y, z) on the sphere and (w_1, w_2) on the plane, the projection is given by

$$(w_1, w_2) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right).$$

Next, compute \dot{R}_1 in this new system of coordinates. For that purpose we parametrize the circle of rotation of some point near the south pole, corresponding to $y = y_0$ constant, as in Figure 3.1. Namely

$$\left(\sqrt{1-y_0^2} \sin(\alpha), y_0, \sqrt{1-y_0^2} \cos(\alpha) - 1 \right),$$

then \dot{R}_1 corresponds to derivative with respect to α , which, in stereographic coordinates is given by

$$\left(\frac{\sqrt{1-y_0^2} \sin(\alpha)}{2 - \sqrt{1-y_0^2} \cos(\alpha)}, \frac{y_0}{2 - \sqrt{1-y_0^2} \cos(\alpha)} \right).$$

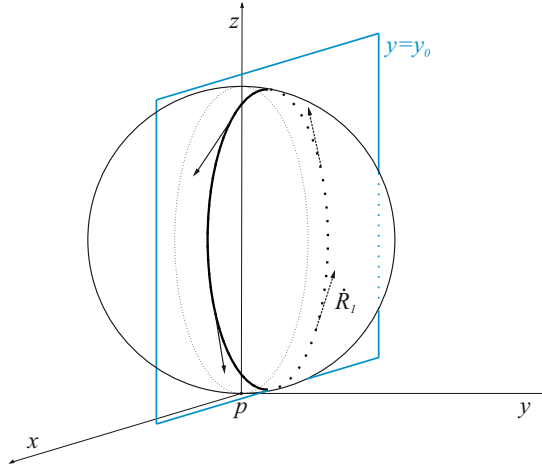


Fig. 3.1.: Rotations on the sphere

Straightforward differentiation and use of Taylor approximations of the functions therein implies

$$\dot{R}_1 = (1 + O(h^2))\partial_{w_1} + O(h^2)\partial_{w_2},$$

for any $y_0, \alpha \leq h$. Similarly for \dot{R}_2 ,

$$\dot{R}_2 = (O(h^2))\partial_{w_1} + O(1 + h^2)\partial_{w_2}.$$

Let us denote by a^{ij} the coefficients of the change of coordinates. As a consequence \dot{R}_i , for $i = 1, 2$ coincides with ∂_{w_i} up to an error of second order. Finally, one have to compute the metric tensor in this coordinates, which we denote by g_{ij} , and the same method shows that it is a perturbation of second order of the identity, i.e. $g_{ij}(y) = \delta_{ij}(x) + O(d_g(x, y)^2)$. This fact will be very convenient in order to apply our previous Lemma 3.1 effectively. One may observe also that the stereographical projection coordinates are not far from being equal to the polar coordinates. That is, they differ only on a second order perturbation, allowing to transfer many estimates and properties from one to the other.

3.2 Nonlinear lower bounds

In this section we provide some specific nonlinear lower bounds for the fractional laplacian and use them to prove the global regularity of the critical surface quasi-geostrophic equation for the two dimensional sphere following a strategy due to Constantin and Vicol [CV12; Con+15]. Their work takes place in euclidean space \mathbb{R}^n as well as in the periodic torus \mathbb{T}^n . This was later extended to the case of bounded domains by Constantin and Ignatova [CI16]. The nonlinear bounds are based on an refinement of the Córdoba-Córdoba inequality [CC04]. However since we are interested in the case of the sphere (and in general compact manifolds) some extra

hypothesis have to be imposed in the statement together with the curvature effects in the error term.

Proposition 3.1. *Let f be a smooth function on \mathbb{S}^2 and $0 < \alpha < 2$. Then, provided $|\nabla_g f(x)| \geq C\|f\|_{L^\infty}$, we have the pointwise bound*

$$\nabla_g f(x) \cdot \nabla_g \Lambda_g^\alpha f(x) \geq \frac{1}{2} \Lambda_g^\alpha(|\nabla_g f|^2)(x) + \frac{1}{4} D(x) + \frac{|\nabla_g f(x)|^{2+\alpha}}{c\|f\|_\infty^\alpha} + O(\|\nabla_g f\|_{L^\infty}^2), \quad (3.4)$$

where $D(x)$ is a positive functional defined in the proof and the constants C, c depend only on fixed quantities, but are independent of x .

Remark 3.3. *The proof also works for the n -dimensional sphere, however since at the end we will just proof the result in the two dimensional case, we just state it for this case. The idea of the proof is to employ the representation formula (3.1) and use the commutator (3.3) in order to obtain positivity in the principal term. We do take advantage of the natural isometries of the round spheres. Direct use of microlocal analysis does not seem to help much at this point.*

Remark 3.4. *The constants appearing in Proposition 3.1, depend on several functions, which are uniformly bounded or are just fixed parameters.*

Proof of Proposition 3.1. Fix the point $x \in \mathbb{S}^2$ and around it consider the stereographical coordinates introduced in the previous section. Then we have $g^{ij}(x) = \delta^{ij}(x)$ and one may compute using local coordinates the left hand side of the inequality (3.4) as follows

$$\begin{aligned} g^{ij}(x) \partial_i f(x) g_{jk}(x) g^{k\ell}(x) \partial_\ell \Lambda_g^\alpha f(x) &= \partial_i f(x) g^{i\ell}(x) \partial_\ell \Lambda_g^\alpha f(x) \\ &= \dot{R}_i f(x) a^{i\ell}(x) \dot{R}_\ell \Lambda_g^\alpha f(x) \\ &= \dot{R}_i f(x) a^{i\ell}(x) (\Lambda_g^\alpha \dot{R}_\ell f)(x) \end{aligned}$$

for $i, \ell = 1, 2$. Therefore, we have that

$$\nabla_g f(x) \cdot \nabla_g \Lambda_g^\alpha f(x) = \nabla f(x) \cdot \nabla \Lambda_g^\alpha f(x),$$

since the distortion $a^{il}(x)$, which is a perturbation of second order of the metric, is not noticed at x . Let us introduce for simplicity the following notation $\nabla f := (\dot{R}_1 f, \dot{R}_2 f)$. The integral representation formula (3.1) for the fractional laplacian yields

$$\nabla f(x) \cdot \Lambda_g^\alpha \nabla f(x) = \frac{1}{2} \Lambda_g^\alpha(|\nabla f|^2)(x) + \frac{1}{2} D(x) + E(x) + O(\|\nabla f\|_{L^\infty}^2),$$

where

$$D(x) = c_\alpha P.V. \int_{\mathbb{S}^2} \frac{|\nabla f(x) - \nabla f(y)|^2}{d_g(x, y)^{2+\alpha}} u_0(x, y) \chi(x, y) d\text{vol}_g(y),$$

and the error term

$$E(x) = \frac{1}{2} \int_{\mathbb{S}^2} \frac{|\nabla f(x) - \nabla f(y)|^2}{d_g(x, y)^{2+\alpha}} k_N(x, y) d\text{vol}_g(y).$$

Since the error term $E(x)$ is less singular seen as an integral operator, one would like to absorb it in, say, $\frac{1}{8}D(x)$, which will be possible due to the regularizing nature of k_N . Indeed, it is clear from the construction of the parametrix that $k_N(x, y)$ is supported where $\chi(x, y)$ is supported and it is bounded a priori by some constant, say, C_χ independently of χ but depending on N . As a consequence one may choose the cut off χ to be supported in $d_g(x, y) \leq \frac{1}{8}C_\chi$ providing the desired estimate. Therefore, we can subsume that term in $D(x)$ after changing conveniently the constant. Next, using the commutator estimate (3.3) and an appropriate cut off function we have that

$$\Lambda_g^\alpha(|\nabla f|^2)(x) - \Lambda_g^\alpha(|\nabla_g f|^2)(x) = O(\|\nabla_g f\|_{L^\infty}^2).$$

Taking all of this into account, we would be done if we prove the lower bound

$$D(x) \geq \frac{|\nabla_g f(x)|^{2+\alpha}}{c\|f\|_{L^\infty}^\alpha}. \quad (3.5)$$

To do that let us introduce a smooth cut-off η_ρ supported outside a ball of radius 2ρ equal to one, and equal to zero inside the ball of radius ρ around x . We will optimize the radius ρ , provided that it is smaller than the radius of the sphere R , to get the desired inequality. Let us observe now that

$$D(x) \geq c_\alpha \int_{\mathbb{S}^2} \frac{|\nabla f(x) - \nabla f(y)|^2}{d_g(x, y)^{2+\alpha}} \eta_\rho(y) u_0(x, y) \chi(x, y) d\text{vol}_g(y).$$

Next we use the fact that for every y we have:

$$|\nabla f(x) - \nabla f(y)|^2 \geq |\nabla f(x)|^2 - 2\nabla f(x) \cdot \nabla f(y).$$

Hence we get

$$\begin{aligned} D(x) &\geq c_\alpha |\nabla f(x)|^2 \int_{\mathbb{S}^2} \frac{\eta_\rho(y) u_0(x, y) \chi(x, y)}{d_g(x, y)^{2+\alpha}} d\text{vol}_g y - c_\alpha |\nabla f(x)| \left| \int_{\mathbb{S}^2} \frac{\nabla f(y) \eta_\rho(y) \chi(x, y)}{d_g(x, y)^{2+\alpha}} d\text{vol}_g(y) \right| \\ &\geq c_\alpha |\nabla f(x)|^2 \int_{\mathbb{S}^2} \frac{\eta_\rho(y) u_0(x, y) \chi(x, y)}{d_g(x, y)^{2+\alpha}} d\text{vol}_g y - c_\alpha |\nabla f(x)| \|f\|_{L^\infty} \int_{\mathbb{S}^2} \left| \nabla \left(\frac{\eta_\rho(y) \chi(x, y)}{d_g(x, y)^{2+\alpha}} \right) \right| \\ &\geq C_1 |\nabla_g f(x)|^2 \left(\frac{1}{\rho^\alpha} - \frac{1}{R^\alpha} \right) - C_2 \frac{|\nabla_g f(x)| \|f\|_{L^\infty}}{\rho^{\alpha+1}} \\ &\geq C_3 \frac{|\nabla_g f(x)|^2}{\rho^\alpha} - C_2 \frac{|\nabla_g f(x)| \|f\|_{L^\infty}}{\rho^{\alpha+1}}, \end{aligned}$$

where $C_1 = C_1(\alpha, \chi, u_0, \mathbb{S}^2)$, $C_2 = C_2(\alpha, \chi, \mathbb{S}^2)$ and $C_3 = C_3(\alpha, \chi, u_0, \mathbb{S}^2, R)$ where we have imposed the size restriction in the cut-off $\rho < \frac{R}{C}$. We would like to set

$$\rho = \frac{C_2 \|f\|_{L^\infty}}{2C_3 |\nabla_g f(x)|},$$

in order to obtain the nonlinear bound (3.5), with $c = \frac{(2C_3)^{1+\alpha}}{c_2^\alpha}$. But due to the size restriction ρ , we can only do this provided

$$|\nabla_g f(x)| \geq C_4 \|f\|_{L^\infty},$$

where $C_4 = C_4(C_1, C_2, C_3)$. □

3.3 Global gradient control of the solution

The aim of this section is to provide a proof of the following result, dealing with a global in time gradient control of the solution to (2.1).

Proposition 3.2. *Let $\theta(x, t)$ be a weak solution of (2.1), then it satisfies the bound*

$$\sup_{t \geq t_0} \|\nabla_g \theta\|_{L^\infty(\mathbb{S}^2)} \leq C \left(\|\theta_0\|_{L^\infty(\mathbb{S}^2)}, \|\nabla_g \theta_0\|_{L^\infty(\mathbb{S}^2)}, \epsilon, \chi, R, u_0 \right). \quad (3.6)$$

Before starting with the proof, let us make some remarks:

Remark 3.5. *During the proof we will assume that θ is smooth function in $[0, T)$, otherwise we could introduce some artificial hyper-regularizing term $\nu \Delta_g \theta$ and proving that the estimates do not depend on ν . However, when studying the evolution of $|\nabla_g \theta|^2(x, t)$ a term of the form $\nabla_g \Delta_g \theta(x, t)$ appears, which a priori, could be troublesome. Fortunately, there is a way to overcome this difficulty, by expressing the gradient as a combination of the form $a^{ij}(x) \dot{R}_j$. The rotations commute with the Laplace-Beltrami operator and the commutator will have terms involving first derivatives of $a^{ij}(x)$, which vanish, and second derivatives of $a^{ij}(x)$, which are uniformly bounded, coupled with (local) first derivatives of $\theta(x, t)$, which can be absorbed by $|\nabla_g \theta|(x, t)$ using the commutator estimate (3.3).*

Remark 3.6. *Notice also that a priori the limit functions might correspond to different weak solutions but one can prove that any two weak solutions coincide provided that one of them is smooth (which we know a posteriori!). Another fact we will be building on is that an a priori estimate on $\|\nabla_g \theta(\cdot, t)\|_{L^\infty}$ immediately implies that θ is smooth for all times, proving Theorem 1.3. This rather elementary facts are included for the sake of completeness in Appendix B.2 and Appendix B.3.*

Remark 3.7. *Recall that by Theorem 1.1, we know that the solution is uniformly continuous for times $t \geq t_0 > 0$. For smaller times, we use the local existence result of solutions to (2.1) which follow by energy estimates (cf. Appendix B.1).*

Proof of Proposition 3.2. Applying ∇_g to equation (2.1) and taking the scalar product with $\nabla_g \theta$ yields

$$\frac{1}{2}(\partial_t + u \cdot \nabla_g)|\nabla_g \theta|^2 + \nabla_g \theta \cdot \nabla_g \Lambda_g \theta + \nabla_g u : \nabla_g \theta \cdot \nabla_g \theta = 0. \quad (3.7)$$

Now, applying the pointwise lower bound of Proposition 3.1 to (3.7) we get

$$\frac{1}{2}L(|\nabla_g \theta|^2)(x) + \frac{1}{4}D(x) + \frac{|\nabla_g \theta(x)|^3}{c\|\theta\|_{L^\infty}} \leq O(|\nabla_g u(x)||\nabla_g \theta(x)|^2) + O(\|\nabla_g \theta\|_{L^\infty}^2), \quad (3.8)$$

where the operator L is given by

$$L(f) = \partial_t f + u \cdot \nabla_g f + \Lambda_g f,$$

and c is a positive universal constant, as in Proposition 3.1. We are omitting the time variable dependence for exposition's clearness. We claim that an estimate of the form

$$\frac{1}{2}L(|\nabla_g \theta|^2)(x) + \frac{|\nabla_g \theta(x)|^3}{c\|\theta\|_{L^\infty}} \leq C|\nabla_g \theta(x)|^2 + O(\|\nabla_g \theta\|_{L^\infty}^2), \quad (3.9)$$

is valid for some constant $C = C(\epsilon, \chi, \|\theta\|_{L^\infty})$.

Formally, the result then follows intuitively when reading (3.9) at a point where $|\nabla_g \theta(x)|^2$ attains its maximum. Indeed, as a consequence of the positivity of the fractional laplacian (cf. Lemma 3.2), the fact that the gradient vanishes when evaluated on a maximum point and using that the cubic power absorbs the right hand side of (3.9), one expects that $\frac{d}{dt}|\nabla_g \theta|^2(x, t) \leq 0$ obstructing its infinite growth. However, notice that to invoke the nonlinear lower bound of Proposition (3.1) we need to suppose that $|\nabla_g \theta|(x) \geq C\|\theta\|_{L^\infty}$. In case this is not true, we would have directly that $\|\nabla_g \theta\|_\infty \leq C\|\theta\|_\infty$, which is even a better bound. We postpone the rigorous argument of this heuristical claim to the end of the proof. The rest of the section will be devoted to achieve this goal, namely the estimate (3.9).

Since we have to bound the right hand side of (3.8), the next step is to show an integral representation of the main term of the velocity $u(x) = \nabla_g^\perp \Lambda_g^{-1} \theta$ similar to the euclidean one (cf. [CV12]). We will take advantage of the sphere geometry but we believe that it could be done in a more general setting. First one may express in local coordinates around x the derivative $\nabla_g u(x)$ as

$$\begin{aligned} \partial_\ell u_i(x) &= \partial_\ell (g_\perp^{ij} \partial_j \Lambda_g^{-1} \theta)(x) = a_{\ell k}(x) \dot{R}_k (g_\perp^{ij} a_{jn} \dot{R}_n \Lambda_g^{-1} \theta)(x) \\ &= a_{\ell k}(x) g_\perp^{ij}(x) a_{jm}(x) \Lambda_g^{-1} \dot{R}_k \dot{R}_m \theta(x) + a_{\ell k}(x) \dot{R}_k g_\perp^{ij}(x) a_{jm}(x) \Lambda_g^{-1} \dot{R}_m \theta(x), \end{aligned}$$

where $l, k, j, m = 1, 2$. Here we work again with the stereographical local coordinates and rewrite them using the rotations \dot{R}_1, \dot{R}_2 introduced in Section 3.1.

An important property of rotations, is that isometries commute with Λ_g . Next, using the integral representation (3.2) for $\Lambda_g^{-1}\theta$ we have that

$$\begin{aligned}\partial_\ell u_i(x) &= a_{\ell k}(x) g_\perp^{ij}(x) a_{jm}(x) \int_{\mathbb{S}^2} \frac{\dot{R}_k \dot{R}_m \theta(y)}{d_g(x, y)} (u_0 \chi + k_N)(x, y) d\text{vol}_g(y) \\ &+ a_{\ell k}(x) \dot{R}_k \left(g_\perp^{ij}(x) a_{jm}(x) \right) \int_{\mathbb{S}^2} \frac{\dot{R}_m \theta(y)}{d_g(x, y)} (u_0 \chi + k_N)(x, y) d\text{vol}_g(y) + \mathcal{O}(\|\theta\|_{L^\infty}) \\ &:= I_1(x) + I_2(x) + \mathcal{O}(\|\theta\|_{L^\infty}),\end{aligned}$$

where both integrals are understood as a principal value. The second integral $I_2(x)$ can be splitted smoothly in two pieces: near points, say $d_g(x, y) \leq \gamma$ at distance γ ; and far away points $d_g(x, y) > \gamma$. Then,

$$\begin{aligned}I_2(x) &= C_g \int_{d_g(x, y) \leq \gamma} \frac{\dot{R}_m \theta(y)}{d_g(x, y)} (u_0 \chi + k_N)(x, y) d\text{vol}_g(y) \\ &+ C_g \int_{d_g(x, y) > \gamma} \frac{\dot{R}_m \theta(y)}{d_g(x, y)} (u_0 \chi + k_N)(x, y) d\text{vol}_g(y) := I_{2,1}(x) + I_{2,2}(x),\end{aligned}$$

where $C_g = a_{\ell k}(x) \dot{R}_k \left(g_\perp^{ij}(x) a_{jm}(x) \right)$. The term $I_{2,1}(x)$ can be bounded directly

$$|I_{2,1}(x)| \leq C \|\nabla_g \theta\|_{L^\infty} \int_0^\gamma d\gamma' + \text{l.o.t} \leq C\gamma \|\nabla_g \theta\|_{L^\infty},$$

since χ, u_0 can be bounded by a constant, as well as, C_g due to the smooth metric and compactness. The l.o.t. (lower order terms) can be trivially bounded, since the function $k_N(x, y) = \mathcal{O}(d_g(x, y))$ regularize the order of the singular integral operator. The term $I_{2,2}(x)$, can be bounded by integrating by parts

$$|I_{2,2}(x)| \leq C \log(\gamma) \|\theta\|_{L^\infty},$$

which might be a rather large constant. Hence,

$$|I_2(x)| \leq C (\gamma \|\nabla_g \theta\|_{L^\infty} + \log(\gamma) \|\theta\|_{L^\infty}).$$

Next, let us estimate the more singular term I_1 , which after integration by parts is given by

$$I_1(x) = \int_{\mathbb{S}^2} \dot{R}_m \theta(y) \dot{R}_k \left(\frac{u_0(x, y) \chi(x, y) + k_N(x, y)}{d_g(x, y)} \right) d\text{vol}_g(y) := I_1^{\text{in}}(x) + I_1^{\text{med}}(x) + I_1^{\text{out}}(x),$$

where we split smoothly the integral into three summands, namely: an inner piece for near points $d_g(x, y) < \rho$, for some specific $\rho(x) > 0$ to be choosen later, a middle piece $\rho < d_g(x, y) < \epsilon$ (where $\epsilon > 0$ will also be choosen later) and an outer part for points $d_g(x, y) > \epsilon$.

To bound, the inner piece, we notice that using the stereographic coordinates and taking advantage of the cancellations on spheres around x , we achieve

$$I_1^{\text{in}}(x) = \int_{d_g(x,y) < \rho} \frac{|\dot{R}_k \theta(y) - \dot{R}_k \theta(x)|}{d_g(x,y)^2} u_0(x,y) \chi(x,y) d\text{vol}_g(y) + \text{l.o.t} := I_{1,1}^{\text{in}}(x) + \text{l.o.t}$$

where the lower order terms can be bounded easily by $\mathcal{O}(\rho(x) \|\nabla_g \theta\|_{L^\infty})$. The main term above, can be estimate by using the Cauchy-Schwarz inequality

$$\begin{aligned} I_{1,1}^{\text{in}}(x) &\leq C \left(\int_{d_g(x,y) < \rho} \frac{|\dot{R}_k \theta(y) - \dot{R}_k \theta(x)|^2}{d_g(x,y)^3} u_0(x,y) \chi(x,y) d\text{vol}_g(y) \right)^{\frac{1}{2}} \left(\int_{d_g(x,y) < \rho} \frac{1}{d_g(x,y)} d\text{vol}_g(y) \right)^{\frac{1}{2}} \\ &\leq C \sqrt{D(x)} \sqrt{\rho(x)}. \end{aligned}$$

The medium piece is given by

$$I_1^{\text{med}}(x) = \int_{\rho < d_g(x,y) < \epsilon} \dot{R}_k(\theta(y) - \theta(x)) \dot{R}_m \left(\frac{u_0(x,y) \chi(x,y)}{d_g(x,y)} \right) d\text{vol}_g(y) + \text{l.o.t} := I_{1,1}^{\text{med}}(x) + \text{l.o.t},$$

where we understand the integration region as smooth cut-off adapted to balls of radius ρ and ϵ . The cutt off function has non zero slope between $\frac{1}{2}\rho$ and ρ and between ϵ and $\frac{3}{2}\epsilon$. Invoking now the modulus of logarithmic modulus of continuity obtained by Theorem 1.1, this yields

$$|\theta(y) - \theta(x)| \leq \delta \text{ for } d_g(x,y) < \epsilon, \quad (3.10)$$

where $\delta > 0$ should be taken in a proper way later on, which is in fact possible by choosing ϵ sufficiently small. Notice that the modulus of continuity was not uniformly for all times, but for $t \geq t_0 > 0$. Therefore, even though we are not writting the time dependence variable, we can only prove the result uniformly in time for $t \geq t_0 > 0$. Integration by parts and using property (3.10), we have that

$$I_1^{\text{med}}(x) \leq C_2 \frac{\delta}{\rho} + \mathcal{O}(\rho \|\nabla_g \theta\|_{L^\infty}),$$

with $C_2 = C_2(\epsilon, \chi, u_0, \mathbb{S}^2)$. The outer piece can be easily bounded using integration by parts by

$$I_1^{\text{out}}(x) \leq C_3,$$

where $C_3 = C_3(\chi, \|\theta\|_{L^\infty}, \epsilon, R)$. Therefore, we can estimate the first term in the right hand side of (3.8) like

$$\begin{aligned} |\nabla_g u(x)| |\nabla_g \theta(x)|^2 &\leq |\nabla_g \theta(x)|^2 I_1^{\text{in}}(x) + |\nabla_g \theta(x)|^2 I_1^{\text{mid}}(x) + |\nabla_g \theta(x)|^2 I_1^{\text{out}}(x) \\ &+ |\nabla_g \theta(x)|^2 I_2 + |\nabla_g \theta(x)|^2 \mathcal{O}(\|\theta\|_{L^\infty}) \end{aligned}$$

Choosing $\rho(x) = \frac{C}{\|\theta\|_{L^\infty} |\nabla_g \theta(x)|}$ for a suitable constant C , and using Young's inequality we obtain

$$\begin{aligned} I_1^{\text{in}}(x) |\nabla_g \theta(x)|^2 &\leq C_1 \sqrt{D(x)} \sqrt{\rho} |\nabla_g \theta(x)|^2 + \mathcal{O}(\rho(x) \|\nabla_g \theta\|_{L^\infty}) |\nabla_g \theta(x)|^2 \\ &\leq \frac{1}{8} D(x) + \frac{|\nabla_g \theta(x)|^3}{c \|\theta\|_{L^\infty}} + \mathcal{O}(\|\nabla_g \theta\|_{L^\infty}). \\ &\leq \frac{1}{4} D(x) + \mathcal{O}(\|\nabla_g \theta\|_{L^\infty}). \end{aligned}$$

For the medium part, it is enough to take $\delta = \frac{C}{\|\theta\|_{L^\infty}^2}$ for a constant proper C so that it does not exceed the half of the cubic term in (3.8),

$$\begin{aligned} I_1^{\text{med}}(x) |\nabla_g \theta(x)|^2 &\leq C_2 \frac{\delta}{\rho} |\nabla_g \theta(x)|^2 + \mathcal{O}(\|\nabla_g \theta\|_{L^\infty}^2) \\ &\leq \frac{|\nabla_g \theta(x)|^3}{2c \|\theta\|_{L^\infty}} + \mathcal{O}(\|\nabla_g \theta\|_{L^\infty}^2). \end{aligned}$$

The outer part,

$$I_1^{\text{out}}(x) |\nabla_g \theta(x)|^2 \leq C_3 |\nabla_g \theta(x)|^2.$$

Finally, taking γ much smaller then the constant c of the cubic term in (3.8):

$$\begin{aligned} I_2(x) |\nabla_g \theta(x)|^2 &\leq |\nabla_g \theta(x)|^2 (\gamma \|\nabla_g \theta\|_{L^\infty} + \log(\gamma) \|\theta\|_{L^\infty}) \\ &\leq \frac{|\nabla_g \theta(x)|^3}{4c \|\theta\|_{L^\infty}} + \mathcal{O}(\|\nabla_g \theta\|_{L^\infty}^2). \end{aligned}$$

Therefore, we have proved that he claimed inequality (3.9)

$$\frac{1}{2} L(|\nabla_g \theta|^2)(x) + \frac{|\nabla_g \theta(x)|^3}{c \|\theta\|_{L^\infty}} \leq C |\nabla_g \theta|^2(x) + \mathcal{O}(\|\nabla_g \theta\|_{L^\infty}^2). \quad (3.11)$$

Evaluating (3.11) at a point \bar{x} , where the maximum of $|\nabla_g \theta(x)|^2$ is attained, we have that

$$\frac{1}{2} \partial_t |\nabla_g \theta(\bar{x}, t)|^2 + \frac{|\nabla_g \theta(\bar{x}, t)|^3}{c \|\theta\|_{L^\infty}} \leq 0,$$

whenever

$$|\nabla_g \theta(\bar{x}, t)| \geq \frac{C}{c \|\theta\|_{L^\infty}} = C_\star.$$

Before, giving the last argumen let us notice that we could change $\|\theta\|_{L^\infty}$ by $\|\theta_0\|_{L^\infty}$ everywhere in the proof by the maximum principle property (cf. [CC04]) which states

$$\|\theta\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}.$$

To conclude the argument, let $K = \max\{C \|\theta_0\|_{L^\infty}^2, \|\nabla_g \theta_0\|_{L^\infty}^2, C_\star^2\}$. This choice of K is enough to show that $|\nabla_g \theta|^2(x, t) \leq K$ for all times $t \geq t_0$ and $x \in \mathbb{S}^2$, a fact that will be proved by contradiction. Let $x(t) \in \mathbb{S}^2$ be the point where $|\nabla_g \theta|^2(\cdot, t)$ attains

its maximum (this is well defined due to continuity and compactness). Even though $t \mapsto x(t)$ is not necessarily continuous, $t \mapsto |\nabla_g \theta|^2(x(t), t)$ is. As a consequence

$$t_0 = \inf\{t \in (0, \infty) : |\nabla_g \theta|^2(x(t), t) \geq K\} \text{ is positive.}$$

We want to prove that $t_0 = \infty$. If not, there exists some finite $t_0 > 0$ for which, by continuity, $|\nabla_g \theta|^2(x(t_0), t_0) \geq K$. But by definition one also knows $|\nabla_g \theta|^2(x(t_0), t) < K$ for any $t < t_0$. This facts altogether imply

$$\frac{d}{dt} |\nabla_g \theta|^2(x(t_0), t_0) \geq 0,$$

which contradicts the inequality above read at the maximum $(x(t_0), t_0)$. \square

Stability, well-posedness and blow-up criterion for the Incompressible Slice Model

The last chapter of the thesis, deals with solution properties of the ISM. We will characterize the equilibrium of solutions as stated in Theorem 1.4, and then study formal (Theorem 1.5) and non linear stability (Theorem 1.6). Afterwards we will focus in the local existence of smooth solutions (Theorem 1.7) and provide a blow-up criteria (Theorem 1.8).

4.1 Functional setting, preliminaries and basic notation

In this section, we will provide the functional setting, fix the notation, as well as presenting the tools that we will use later on to prove the main theorems. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. For $\alpha \in \mathbb{N}^n$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, and $f \in C^\infty(\Omega)$, we employ the multi-index notation

$$D^\alpha f = D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_n} f,$$

and denote $|\alpha| = \sum_{i=1}^n |\alpha_i|$. For any integer $s \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$, we define the Sobolev norm

$$\|f\|_{W^{s,p}(\Omega)} = \sum_{\alpha \in \mathbb{N}^n: |\alpha| \leq s} \|D^\alpha f\|_{L^p(\Omega)}.$$

Let us define the Sobolev space $W^{s,p}(\Omega)$ as the closure of C^∞ functions with compact support with respect to the norm $W^{s,p}(\Omega)$. When the spaces are L^2 -based, they also turn out to be Hilbert and are denoted by $H^s(\Omega) = W^{s,2}(\Omega)$, with the interior product

$$(f, g)_{H^s(\Omega)} = \sum_{\alpha \in \mathbb{N}^n: |\alpha| \leq s} \int_{\Omega} D^\alpha f D^\alpha g \, dV.$$

Let us introduce some important well-known calculus inequalities [Bea+84], [KM81]:

Lemma 4.1. (i) If $f, g \in H^s(\Omega) \cap C(\Omega)$, then

$$\|fg\|_{H^s} \leq C_{s,n}(\|f\|_{L^\infty} \|D^s g\|_{L^2} + \|D^s f\|_{L^2} \|g\|_{L^\infty}).$$

(ii) If $f \in H^s \cap C^1(\Omega)$ and $g \in H^{s-1} \cap C(\Omega)$, then for $|\alpha| \leq s$,

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2} \leq C'_{s,n}(\|f\|_{W^{1,\infty}}\|g\|_{H^{s-1}} + \|f\|_{H^s}\|g\|_{L^\infty}).$$

To estimate and deal with some boundary terms later on, we will invoke the so called Trace Theorem [LM72], [Aub82].

Theorem 4.1. *Let $u \in W^{s,p}(\Omega)$, then there exist constants $C_{n,p,s} > 0$ such that*

$$\|u\|_{W^{s-\frac{1}{p},p}(\partial\Omega)} \leq C_{n,p,s}\|u\|_{W^{s,p}(\Omega)}.$$

Now we introduce some functional spaces used throughout the chapter. Let

$$H_\star^m = \{u \in H^m(\Omega) : \operatorname{div} u = 0, u \cdot n = 0 \text{ on } \partial\Omega\}$$

for $m \geq 1$, and

$$H_\star^0 = \{u \in L^2(\Omega) : \operatorname{div} u = 0, u \cdot n = 0 \text{ on } \partial\Omega\}.$$

Let us also mention the Helmholtz-Hodge decomposition Theorem and some properties of the Leray's projection operator.

Lemma 4.2. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and w is a vector field defined on Ω . Then we can decompose w in the form*

$$w = u + \nabla p,$$

where

$$\operatorname{div} u = 0, u \cdot n = 0 \text{ on } \partial\Omega, \int_\Omega u \cdot \nabla p \, dV = 0.$$

The operator $\mathbb{P} : w \rightarrow u$ is called the Leray's projector and it has the following properties:

$$(i) \operatorname{div} (\mathbb{P}w) = 0, (\mathbb{P}w) \cdot n = 0 \text{ on } \partial\Omega, \|\mathbb{P}w\|_{L^2(\Omega)} \leq \|w\|_{L^2(\Omega)}.$$

(ii) Set $\mathbb{Q} = 1 - \mathbb{P}$; this is, $\mathbb{Q}w = \nabla p$. Then if $w_1 = u_1 + \nabla p_1$, $w_2 = u_2 + \nabla p_2$, we have

$$(\mathbb{P}w_1, \mathbb{Q}w_2)_{L^2} = (u_1, \nabla p_2)_{L^2} = -(\operatorname{div} u_1, p_2)_{L^2} = 0.$$

Among other things, we are interested in studying formal and nonlinear stability of the equilibrium solutions of the Incompressible Slice Model. We therefore introduce the method we use, which was developed in [Hol+85]. First of all, let us provide the definition of equilibrium solution:

Definition 4.1. Let P be a Banach space of velocity fields u and let $X : P \rightarrow P$ be an operator. This defines a dynamical system by

$$\dot{u} = X(u). \quad (4.1)$$

We say that a velocity profile u_e is an equilibrium point of (4.1) if $X(u_e) = 0$.

In a system like (4.1) one can study different types of stability:

- (i) *Spectral stability.* A system is said to be spectrally stable if the spectrum of its linearization $DX(u_e)$ has no strictly positive real part.
- (ii) *Linearized stability.* A system is said to be linearized stable if its linearization at the equilibrium point u_e is stable.
- (iii) *Formal stability.* A system is said to be formally stable if there exists a conserved quantity such that its first variation vanishes at the equilibrium point u_e , and whose second variation is definite (either positive or negative) at this point.
- (iv) *Nonlinear stability.* We say that an equilibrium solution u_e is nonlinearly stable if there exists a norm $\|\cdot\|$ and for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\|u(0) - u_e\| < \delta$ then $\|u(t) - u_e\| < \epsilon$, for $t > 0$.

Remark 4.1. It is important to study equilibrium solutions of physical systems and their stability. For instance, in the case of atmospheric models, these equilibrium solutions represent steady states of the atmospheric vector fields. Studying stability around steady states provides us with insight on whether some of these states can be destroyed in the short-term by small perturbations, or on the contrary, are expected to remain rather stable.

Next, let us explain the Energy-Casimir algorithm presented in [Hol+85] for the study of formal and nonlinear stability of equilibrium solutions of a dynamical system. It consists of six steps:

- (i) Consider a system of the type (4.1). The first step consists in finding an integral of motion H for this system. Often, this equation can be expressed in terms of a Poisson bracket $\{\cdot, \cdot\}$.
- (ii) Find a parametric family of constants of motion C_Φ , where Φ belongs to some general class of functions. This is, these functions need to satisfy

$$\frac{D}{Dt} C_\Phi = 0,$$

where $\frac{D}{Dt} := \partial_t + u \cdot \nabla$ stands for the material derivative. One way of doing this is to look for Casimirs of the Poisson bracket, which are functions C such that $\{\cdot, C\} = 0$.

(iii) Construct a generalized conserved quantity

$$H_\Phi = H + C_\Phi.$$

Impose $DH_\Phi(u_e) = 0$, yielding a condition on Φ .

(iv) Find quadratic forms Q_1, Q_2 on P such that

$$\begin{aligned} Q_1(\Delta u) &\leq H(u_e + \Delta u) - H(u_e) - DH(u_e) \cdot \Delta u, \\ Q_2(\Delta u) &\leq C_\Phi(u_e + \Delta u) - C_\Phi(u_e) - DC_\Phi(u_e) \cdot \Delta u, \end{aligned}$$

for $\Delta u \in P$. Moreover, require that

$$Q_1(\Delta u) + Q_2(\Delta u) > 0.$$

(v) Obtain an estimate

$$Q_1(u(t) - u_e) + Q_2(u(t) - u_e) \leq H_\Phi(u(0)) - H_\Phi(u_e),$$

often expressed as

$$Q_1(\Delta u(t)) + Q_2(\Delta u(t)) \leq H_\Phi(u(0)) - H_\Phi(u_e).$$

(vi) Define a norm on P by

$$\|u\| := Q_1(u) + Q_2(u).$$

Theorem 4.2 ([Arn89; Hol+85]). *Assume steps (i)-(vi) are satisfied. If H_Φ is continuous at u_e on the norm $\|\cdot\|$, and solutions of (4.1) exist for all time, then u_e is a nonlinearly stable equilibrium point.*

Remark 4.2. *A sufficient condition for the continuity of H_Φ is the existence of positive constants C_1, C_2 such that*

$$\begin{aligned} H(u_e + \Delta u) - H(u_e) - DH(u_e) \cdot \Delta u &\leq C_1 \|\Delta u\|^2, \\ C_\Phi(u_e + \Delta u) - C_\Phi(u_e) - DC_\Phi(u_e) \cdot \Delta u &\leq C_2 \|\Delta u\|^2. \end{aligned}$$

Finally, we also wish to formulate an abstract theorem by Kato and Lai [KL84] that we will use to prove our local existence and uniqueness result for the ISM equations

(1.36)-(1.39) with boundary condition (1.40). Before stating the theorem, we need to introduce some definitions first.

Definition 4.2. We say that a family $\{V, H, X\}$ of three real separable Banach spaces is an admissible triplet if the following conditions are met:

- $V \subset H \subset X$, with the inclusions being dense and continuous.
- H is a Hilbert space with inner product $(\cdot, \cdot)_H$ and norm $\|\cdot\|_H = (\cdot, \cdot)_H^{\frac{1}{2}}$.
- There is a continuous, nondegenerate bilinear form on $V \times X$, denoted by $\langle \cdot, \cdot \rangle$, such that

$$\langle v, u \rangle = (v, u)_H, \quad v \in V, \quad u \in H.$$

Remark 4.3. Recall that a bilinear form $\langle \cdot, \cdot \rangle$ is continuous if

$$|\langle v, u \rangle| \leq C \|v\|_V \|u\|_X, \quad \text{for some constant } C > 0, v \in V, \text{ and } u \in X.$$

For nondegeneracy we need

$$\langle v, u \rangle = 0 \text{ for all } u \in X \text{ implies } v = 0,$$

$$\langle v, u \rangle = 0 \text{ for all } v \in V \text{ implies } u = 0.$$

Definition 4.3. We say that $A : [0, T] \times H \rightarrow X$ is a sequentially weakly continuous map if $A(t_n, v_n) \rightharpoonup A(t, v)$ in X whenever $t_n \rightarrow t$ and $v_n \rightharpoonup v$ in H .

We denote by $C_w([0, T]; H)$ the space of sequentially weakly continuous functions from $[0, T]$ into H , and by $C_w^1([0, T]; X)$ the space of functions $f \in W^{1, \infty}([0, T]; X)$ such that $\frac{df}{dt} \in C_w([0, T]; X)$. With these notions established, the existence theorem by Kato and Lai reads:

Theorem 4.3. ([KL84]) Consider the abstract nonlinear evolution equation

$$\begin{cases} u_t + A(t, u) = 0, \\ u(0) = \phi, \end{cases} \quad (4.2)$$

where $A(t, u)$ is a nonlinear operator. Let $\{V, H, X\}$ be an admissible triplet. Let A be a weakly continuous map on $[0, T_0] \times H$ into X such that

$$\langle u, A(t, u) \rangle \geq -\beta(\|u\|_H^2), \quad \text{for } t \in [0, T_0], \quad u \in V, \quad (4.3)$$

where $\beta(r) \geq 0$ is a monotone increasing function of $r \geq 0$. Then for any $\phi \in H$, there exists $T \in (0, T_0)$ such that (4.3) has a solution

$$u \in C_w([0, T]; H) \cap C_w^1([0, T]; X).$$

Moreover,

$$\|u(t)\|_H^2 \leq \rho(t), \quad t \in [0, T],$$

where $\rho(t)$ is a continuous increasing function on $[0, T]$.

Remark 4.4. If $A([0, T_0] \times V) \subset H$, we can rewrite (4.3) in a more convenient form, namely,

$$(u, A(t, u))_H \geq -\beta(\|u\|_H^2), \quad t \in [0, T_0], \quad u \in V.$$

Remark 4.5. The existence time T and ρ can be determined by solving the scalar differential equation

$$\begin{cases} \rho_t = 2\beta(\rho), \\ \rho(0) = \|\phi\|_H^2, \end{cases} \quad (4.4)$$

where T is any value that ensures (4.4) has a solution $\rho(t)$ in $[0, T]$. It is important to mention that Theorem 4.3 does not guarantee uniqueness of solutions.

4.2 The Incompressible Slice Model (ISM)

In this section, we provide a review of the Cotter-Holm Slice Model. For this, we consider its Lagrangian particle map, which defines the evolution of a fluid configuration after a time t , and we show that the possible configurations of this fluid system form a Lie group. The natural space for the velocities is the group's Lie algebra. On the Lie algebra, and for a choice of Lagrangian function, we can apply Hamilton's principle and derive evolution equations for the Slice Model dynamics.

4.2.1 The Cotter-Holm Slice Model

In the CHSM [CH13], one considers a dynamical system with Lagrangian evolution map of the form

$$\phi(X, Y, Z, t) = (x(X, Z, t), y(X, Z, t) + Y, z(X, Z, t)), \quad (4.5)$$

for $(X, Z) \in \Omega \subset \mathbb{R}^2$ an open bounded domain, and $Y \in \mathbb{R}$. Here, Ω is called the slice and y is the transverse component to the slice. The change in time of the Eulerian coordinate paths $x(X, Z, t)$ and $z(X, Z, t)$ representing the motion of a fluid parcel in the vertical slice Ω , is assumed to depend only on its Lagrangian coordinates, or labels, (X, Z) in the reference slice configuration. The change in the transverse Eulerian component $y(X, Y, Z, t)$ is taken to depend on the coordinates X, Z , plus a linear variation in Y .

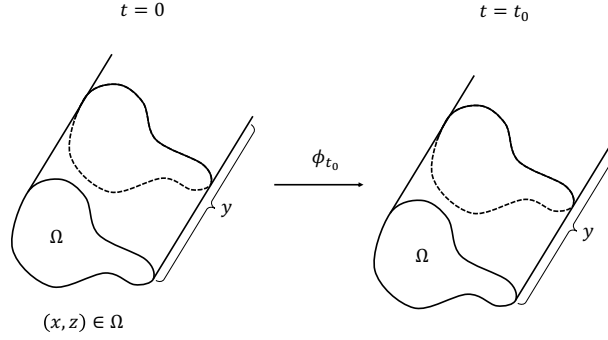


Fig. 4.1.: The Lagrangian map ϕ explains how to move from a fluid configuration at time $t = 0$ to a configuration at time $t = t_0$.

The set of Lagrangian maps of the type (4.5) can be modelled as $G = \text{Diff}(\Omega) \circledast \mathcal{F}(\Omega)$, where $\text{Diff}(\Omega)$ denotes the group of diffeomorphisms in Ω , and $\mathcal{F}(\Omega)$ represents the group of differentiable real functions in Ω . The symbol \circledast denotes semi-direct product between two algebraic groups. G can be endowed with a Lie group structure, with multiplication representing composition of Lagrangian maps (4.5). Multiplication in the group is given by the formula

$$(\phi_1, f_1) * (\phi_2, f_2) = (\phi_2 \circ \phi_1, f_2 \circ \phi_1 + f_1), \quad (4.6)$$

for $\phi_1, \phi_2 \in \text{Diff}(\Omega)$, $f_1, f_2 \in \mathcal{F}(\Omega)$. This makes $(G, *)$ a Lie group. The operation $*$ turns out to be a right action.

Remark 4.6. Formula (4.6) represents the result of the composition of two Lagrangian maps of the type (4.5), so the product on the Lie group describes the dynamics of the particles in the Slice Model.

If a Lie group G represents the motions of a given physical system, the natural space for the velocities is its Lie algebra. The right-invariant Lie algebra of G can be identified with the space $\mathfrak{g} = \mathfrak{X}(\Omega) \circledast \mathcal{F}(\Omega)$, where $\mathfrak{X}(\Omega)$ represents the set of right-invariant vector fields on Ω . This implies the velocity in this model has two components: the slice component $u_S \in \mathfrak{X}(\Omega)$, and the component in the y direction (the transverse component), $u_T \in \mathcal{F}(\Omega)$.

The Lie bracket on \mathfrak{g} for right-invariant vector fields (remember that the action on G is right-invariant) is defined by the formula

$$[(u_S, u_T), (w_S, w_T)] = ([u_S, w_S], u_S \cdot \nabla w_T - w_S \cdot \nabla u_T), \quad u_S, w_S \in \mathfrak{X}(\Omega), \quad u_T, w_T \in \mathcal{F}(\Omega).$$

Advected Eulerian quantities are defined to be the variables which are Lie transported by the Eulerian velocity. Conservation of mass comes from the right action by the

inverse for advected quantities $a(t) = a_0 g^{-1}(t)$, where a_0 denotes the advected quantity at time zero (in this case the mass density), and $g(t) \in G$ denotes the flow (in this case $\phi(t)$). Conservation of mass in the CHSM reads

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{(u_S, u_T)} \right) (D \, \mathrm{d}^3 x) = \left(\frac{\partial D}{\partial t} + \nabla \cdot (u_S D) + \frac{\partial(u_T D)}{\partial y} \right) \mathrm{d}^3 x = 0,$$

where D denotes the mass density, and $\mathcal{L}_{(u_S, u_T)}(D \, \mathrm{d}^3 x)$ is the Lie derivative of the three form $D \, \mathrm{d}^3 x = D(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$. Since D and u_T are assumed to be y -independent, conservation of mass can be reformulated as

$$\partial_t D + \nabla \cdot (u_S D) = 0 \quad (4.7)$$

In this model, potential temperature is defined by

$$\theta(x, y, z, t) = \theta_S(x, z, t) + (y - y_0)s, \quad (4.8)$$

and the tracer equation for the potential temperature, implied by the right action of the inverse flow on advected quantities, becomes

$$\partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T s = 0. \quad (4.9)$$

Remark 4.7. Note that (4.8) is a very special way of defining potential temperature, since it is assumed it varies linearly on the y direction. This proves to be useful for having circulation theorems (see Subsection 4.2.4) which do not hold in the general case.

Regarding the mathematical framework, D is considered as an element in $\wedge^2(\Omega)$, defined to be the space of two form-densities $\sum_{i=1}^m \alpha_i(x, z) \, \mathrm{d}x \, \mathrm{d}z$, and θ_S is an element in $\mathcal{F}(\Omega)$ (a differentiable scalar function). Indeed, this is because equations (4.7) and (4.9) can be rewritten respectively in Lie derivative notation as

$$(\partial_t + \mathcal{L}_{u_S})(D \, \mathrm{d}S) = 0,$$

and

$$(\partial_t + \mathcal{L}_{u_S})\theta_S = -u_T s.$$

The CHSM considers the reduced Lagrangian function

$$l[(u_S, u_T), (\theta_S, s), D] : (\mathfrak{X}(\Omega) \circledast \mathcal{F}(\Omega)) \circledast ((\mathcal{F}(\Omega) \times \mathbb{R}) \times \wedge^2(\Omega)) \rightarrow \mathbb{R}.$$

As already explained, the tracers are advected by the Eulerian flow. The velocity vector field is right-invariant in Eulerian coordinates, so this implies that the Lagrangian function must be right-invariant, and therefore, reduction by symmetry can be performed. The equations of motion in the CHSM are the equations coming from

the variational principle for this general Lagrangian function. This is, we consider the action functional

$$S[(u_S, u_T), (\theta_S, s), D] = \int_0^T l[(u_S, u_T), (\theta_S, s), D] dt,$$

where the last integral is taken over closed paths for the variables, vanishing at the endpoints. Apply Hamilton's principle and obtain

$$0 = \delta S = \delta \int_0^T l[(u_S, u_T), (\theta_S, s), D] dt,$$

from which one can derive the CHSM equations after integration by parts (c.f. [CH13]):

$$\partial_t \left(\frac{\delta l}{\delta u_S} \right) + \nabla \cdot \left(u_S \otimes \frac{\delta l}{\delta u_S} \right) + (\nabla u_S)^T \cdot \frac{\delta l}{\delta u_S} + \frac{\delta l}{\delta u_T} \nabla u_T = D \nabla \frac{\delta l}{\delta D} - \frac{\delta l}{\delta \theta_S} \nabla \theta_S, \quad (4.10a)$$

$$\partial_t \left(\frac{\delta l}{\delta u_T} \right) + \nabla \cdot \left(u_S \frac{\delta l}{\delta u_T} \right) = - \frac{\delta l}{\delta \theta_S} s, \quad (4.10b)$$

$$\partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T s = 0, \quad (4.10c)$$

$$\nabla \cdot u_S = 0. \quad (4.10d)$$

These are the equations for a general unspecified Lagrangian function. In the next subsections, we will substitute a particular Lagrangian, namely, the Lagrangian in the incompressible Euler-Boussinesq case, which has important physical significance.

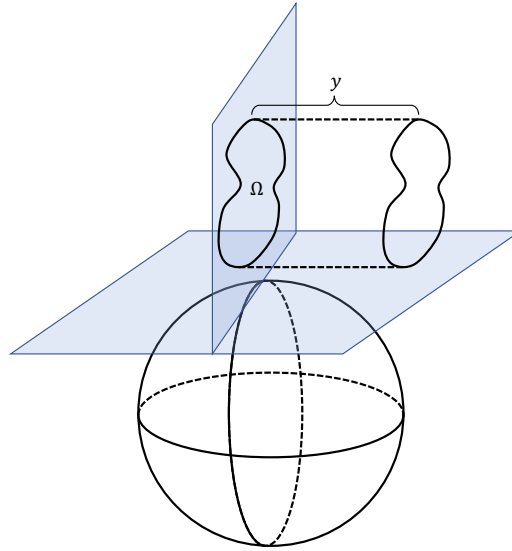


Fig. 4.2.: Fluid motion in the vertical slice Ω coupled dynamically to the flow velocity transverse to the slice.

4.2.2 The CHSM and front formation

As a motivation for defining this model, one can think that slice models are useful to study atmospheric processes where the dependence of the particles on one of the variables can be simply approximated. Following [HB71], atmospheric fronts are generated by changing temperature gradients. More specifically, they are associated with discontinuities in velocity and potential temperature. There are many mechanisms which trigger frontogenesis, like:

- (i) A horizontal deformation field.
- (ii) Horizontal shearing motion.
- (iii) A vertical deformation field.
- (iv) Differential vertical motion.

Mechanism (i) is the classical frontogenesis mechanism postulated by Bergeron [Ber37]. Mechanism (ii) is crucial in the dynamics of frontal systems, and has been studied by Sawyer and Eliassen [Eli59]. Mechanism (iv) can have either frontolytic or frontogenetic effects, and has been found responsible, for instance, for the lack of sharpness of surface fronts in the middle troposphere. Also, mechanisms (i)-(ii) operate on the synoptic scale (large scale geostrophic processes), while (iii)-(iv) are dominant on the scale of the front (and are motions pertaining to the baroclinic flow which give rise to the rapid formation of a discontinuity).

Fronts form on the Earth when there is a strong temperature gradient on the North-South direction. In [HB71], fronts are formulated mathematically. As an approximation, one considers geostrophic balance in the cross-front direction. After formulating the equations of motion and nondimensionalising, one seeks a solution of the type

$$u = -\alpha x + u'(x, z), \quad (4.11)$$

$$v = \alpha y + v'(x, z), \quad (4.12)$$

$$w = w(x, z), \quad (4.13)$$

$$\theta = \theta(x, z), \quad (4.14)$$

which is consistent with the approximation made in [CH13]. Note that (u, v) in (4.11)-(4.14) is represented by the 2D slice velocity u_S in the CHSM, and w is represented by u_T .

4.2.3 The Incompressible Slice Model

For the incompressible Euler-Boussinesq Eady model in a smooth domain $(x, z) \in \Omega$, the Lagrangian function is

$$l[u_S, u_T, D, \theta_S, p] = \int_{\Omega} \left\{ \frac{D}{2} (|u_S|^2 + |u_T|^2) + D f u_T x + \frac{g}{\theta_0} D \theta_S z + p(1 - D) \right\} dV, \quad (4.15)$$

where g is the acceleration due to the gravity, θ_0 is the reference temperature, f is the Coriolis force parameter, which is assumed to be a constant, and p is a multiplier which imposes the constraint $D = 1$, implying $\nabla \cdot u_S = 0$ (incompressibility).

Remark 4.8 (Motivation for the Euler-Boussinesq Lagrangian). *Note that the integral in (4.15) can be expressed as*

$$\int_{\Omega} K E(x, z) dV + \int_{\Omega} P_f E(x, z) dV + \int_{\Omega} I E(x, z) dV + \int_{\Omega} p(1 - D) dV = K E + P_f E + I E,$$

supplemented with constraint $D = 1$. Above, $K E$ represents the kinetic energy, $P_f E$ the work done by the Coriolis force stored as potential energy in the fluid, and $I E$ the internal energy of the system.

The ISM equations are the CHSM equations (4.10a)-(4.10d) for the Lagrangian function (4.15), which can be computed [CH13] to be

$$\partial_t u_S + u_S \cdot \nabla u_S - f u_T \hat{x} = -\nabla p + \frac{g}{\theta_0} \theta_S \hat{z}, \quad (4.16a)$$

$$\partial_t u_T + u_S \cdot \nabla u_T + f u_S \cdot \hat{x} = -\frac{g}{\theta_0} z s, \quad (4.16b)$$

$$\partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T s = 0, \quad (4.16c)$$

$$\nabla \cdot u_S = 0, \quad (4.16d)$$

supplemented with the boundary condition

$$u_S \cdot n = 0 \quad \text{on } \partial\Omega. \quad (4.17)$$

Here \hat{x} is the unit normal in the x-direction and s is a constant which measures the variation of the potential temperature in the direction transverse to the slice (for further concreteness, see [CH13]).

4.2.4 Conserved quantities in the ISM

The CHSM enjoys some conservation laws due to its variational character. Here, we state them in the particular case of the ISM, since it is the model we will be working with throughout the rest of this chapter. The circulation velocity in the ISM is defined by

$$v_S = s u_S - (u_T + f x) \nabla \theta_S.$$

Theorem 4.4 (Circulation conservation [CH13]). *The ISM (4.16a)-(4.16d) conserves the circulation of v_S on loops $c(u_S)$ carried by u_S*

$$\frac{d}{dt} \oint_{c(u_S)} v_S \cdot ds = \oint_{c(u_S)} d\pi = 0. \quad (4.18)$$

Corollary 4.1 (Conservation of potential vorticity [CH13]). *The ISM potential vorticity, defined by $q = \text{curl}(v_S) \cdot \hat{z}$ is conserved on fluid parcels,*

$$\frac{Dq}{Dt} = \partial_t q + u_S \cdot \nabla q = 0. \quad (4.19)$$

Remark 4.9. *Note that the definition of potential vorticity above differs from the usual definition of potential vorticity in fluid dynamics, since we have to take into account the transverse velocity u_T , the Coriolis force f , and the potential temperature θ_S . In other words, we are adding the Ertel potential vorticity $\nabla \theta_S \times \nabla(u_T + fx)$.*

The ISM has also the following integral conserved quantities:

Theorem 4.5 (Conserved energy and enstrophy [CH13]). *The ISM (4.16a)-(4.16d) conserves energy and generalised enstrophy:*

$$h = \int_{\Omega} \left\{ \frac{1}{2} |u_S|^2 + \frac{1}{2} u_T^2 - \gamma_S \theta_S \right\} dV, \quad (\text{Energy}) \quad (4.20)$$

$$C_{\Phi} = \int_{\Omega} \Phi(q) dV, \quad (\text{Generalised enstrophy}) \quad (4.21)$$

for any differentiable function Φ of the potential vorticity $q := \text{curl}(v_S) \cdot \hat{z}$, and $\gamma_S = (g/\theta_0)z$.

Remark 4.10. *Conserved quantities are fundamental to apply the Energy-Casimir algorithm ([Hol+85]) for the study of stability, and serve to guarantee integrability properties of the system.*

4.3 Stability of solutions of the ISM

In this section, we study the formal and nonlinear stability of the ISM equations (4.16a)-(4.16d) via the Energy-Casimir algorithm introduced in Section 4.1.

4.3.1 Characterization of ISM equilibrium solutions: Proof of Theorem 1.4

We first present the proof of Theorem 1.4, which consists in a characterization of a class of equilibrium solutions of the ISM. To that purpose, one constructs a generalised conserved quantity of the type

$$H_{\Phi} = h + C_{\Phi}$$

Remark 4.11. Critical points of $H_\Phi = h + C_\Phi$ are Eady equilibrium solutions of (4.10a)-(4.10d). This is a general property of the Energy-Casimir algorithm. For the proof, see the appendix in [Hol+85].

We construct a generalised conserved quantity by

$$H_\Phi = h + C_\Phi = \int_{\Omega} \left\{ \frac{1}{2}(|u_S|^2 + u_T^2) - \gamma_S \theta_S \right\} dV + \int_{\Omega} \Phi(q) dV + \sum_{i=0}^n a_i \int_{\partial\Omega_i} v_S \cdot ds.$$

Taking the first variation, one obtains

$$\begin{aligned} \delta H_\Phi(\delta u_S, \delta u_T, \delta \theta_S) &= \int_{\Omega} (u_S \cdot \delta u_S + u_T \delta u_T - \gamma_S \delta \theta_S) dV \\ &\quad + \int_{\Omega} \Phi'(q) \delta q dV + \sum_{i=0}^n a_i \int_{\partial\Omega_i} \delta v_S \cdot ds := I_1 + I_2 + I_3. \end{aligned}$$

We have that

$$\begin{aligned} I_2 + I_3 &= \int_{\Omega} \Phi'(q) \operatorname{curl}(\delta v_S) \cdot \hat{z} dV + \sum_{i=0}^n a_i \int_{\partial\Omega_i} \delta v_S \cdot ds \\ &= \int_{\Omega} \operatorname{curl}(\Phi'(q) \hat{z}) \cdot \delta v_S dV - \int_{\Omega} \operatorname{div}(\Phi'(q) \hat{z} \times \delta v_S) dV + \sum_{i=0}^n a_i \int_{\partial\Omega_i} \delta v_S \cdot ds \\ &= \int_{\Omega} \operatorname{curl}(\Phi'(q) \hat{z}) \cdot \delta v_S dV + \sum_{i=0}^n (a_i - \Phi'(q)|_{\partial\Omega_i}) \int_{\partial\Omega_i} \delta v_S \cdot ds, \end{aligned} \quad (4.22)$$

so we obtain $a_i = \Phi'(q_e|_{\partial\Omega_i})$, $i = 1, \dots, n$. Here we have used the well-known calculus formula

$$\operatorname{div}(A \times B) = B \cdot \operatorname{curl} A - A \cdot \operatorname{curl} B, \quad \text{with } A, B \text{ vector fields,}$$

and the Divergence Theorem. We can rewrite the first term in (4.22) as

$$\begin{aligned} \int_{\Omega} \operatorname{curl}(\Phi'(q) \hat{z}) \cdot \delta v_S dV &= \int_{\Omega} \operatorname{curl}(\Phi'(q) \hat{z})_s \cdot \delta u_S dV - \int_{\Omega} \operatorname{curl}(\Phi'(q) \hat{z}) \cdot \nabla \theta_S \delta u_T dV \\ &\quad - \int_{\Omega} \operatorname{curl}(\Phi'(q) \hat{z})(u_T + fx) \cdot \nabla \delta \theta_S dV. \end{aligned} \quad (4.23)$$

Notice that the last term in (4.23)

$$\int_{\Omega} \operatorname{curl}(\Phi'(q) \hat{z})(u_T + fx) \cdot \nabla \delta \theta_S dV = - \int_{\Omega} \operatorname{curl}(\Phi'(q) \hat{z}) \cdot (\nabla u_T + f \hat{x}) \delta \theta_S dV,$$

where we have taken into account that $\operatorname{div} \operatorname{curl} g = 0$, for any smooth vector field g , and we have needed the condition

$$\int_{\Omega} \operatorname{div}(\operatorname{curl}(\Phi'(q) \hat{z})(u_T + fx) \delta \theta_S) dV = \int_{\partial\Omega} (\operatorname{curl}(\Phi'(q) \hat{z})(u_T + fx) \delta \theta_S) \cdot n ds = 0. \quad (4.24)$$

It is easily checked that (4.24) is guaranteed if $\text{curl}(\Phi'(q_e)\hat{z}) \cdot n = 0$ at the boundary, which holds since

$$s \text{curl}(\Phi'(q_e)\hat{z}) = -u_{Se},$$

and due to the boundary condition (4.17). Collecting all our previous computations we have that

$$\begin{aligned} \delta H_\Phi(\delta u_S, \delta u_T, \delta \theta_S) &= \int_\Omega (u_S \cdot \delta u_S + u_T \delta u_T - \gamma_S \delta \theta_S) dV + \int_\Omega \text{curl}(\Phi'(q)\hat{z}) \cdot \delta v_S dV \\ &= \int_\Omega (u_S \cdot \delta u_S + u_T \delta u_T - \gamma_S \delta \theta_S) dV + \int_\Omega \text{curl}(\Phi'(q)\hat{z}) s \cdot \delta u_S dV \\ &\quad - \int_\Omega \text{curl}(\Phi'(q)\hat{z}) \cdot \nabla \theta_S \delta u_T dV + \int_\Omega \text{curl}(\Phi'(q)\hat{z}) \cdot (\nabla u_T + f\hat{x}) \delta \theta_S dV, \end{aligned}$$

obtaining the first part of the theorem.

Remark 4.12. Note that the equations obtained for the variables after imposing the first variation of H_Φ to vanish satisfy the equilibrium conditions for the solutions of the ISM (4.16a)-(4.16d), as explained in Remark 4.11.

Let us now show how to rewrite the ISM equations in curl form at equilibrium, which is useful if we want to characterize its equilibrium solutions in terms of a Bernoulli function. We define $\omega_S = \text{curl}(u_S)$. One can express equation (4.16a) as

$$\frac{\partial u_S}{\partial t} = -\omega_S \times u_S - \nabla(p + |u_S|^2/2) + f u_T \hat{x} + \frac{g}{\theta_0} \theta_S \hat{z}.$$

Collecting all the gradient terms on the right-hand side, we have

$$\begin{aligned} \frac{\partial u_S}{\partial t} &= -\omega_S \times u_S - \nabla(p + |u_S|^2/2) + f u_T \nabla x + \frac{g}{\theta_0} \theta_S \nabla z \\ &= -\omega_S \times u_S - \nabla \left(p + |u_S|^2/2 - f u_T x - \frac{g}{\theta_0} \theta_S z \right) - f x \nabla u_T - \frac{g}{\theta_0} z \nabla \theta_S. \end{aligned} \quad (4.25)$$

We calculate

$$-(\nabla \theta_S \times \nabla(u_T + f x)) \times u_S = u_S \cdot \nabla(u_T + f x) \nabla \theta_S - (u_S \cdot \nabla \theta_S) \nabla(u_T + f x).$$

By using (4.16b) and (4.16c), one can obtain the relation

$$-(\nabla \theta_{Se} \times \nabla(u_{Te} + f x)) / s \times u_{Se} = -\frac{g}{\theta_0} z \nabla \theta_{Se} - f x \nabla u_{Te} + \nabla(u_{Te}^2/2 + u_{Te} f x). \quad (4.26)$$

Now, put together (4.25) and (4.26) to derive

$$-(q_e/s) \times u_{Se} - \nabla \left(p_e + |u_{Se}|^2/2 + u_{Te}^2/2 - \frac{g}{\theta_0} z \theta_{Se} \right) = 0. \quad (4.27)$$

By dotting (4.27) against u_{Se} , one obtains

$$u_{Se} \cdot \nabla \left(p_e + |u_{Se}|^2/2 + u_{Te}^2/2 - \gamma_S \theta_{Se} \right) = 0,$$

which, together with the equation for conservation of potential vorticity (4.19), yields the following Bernoulli condition for the ISM:

$$p_e + |u_{Se}|^2/2 + u_{Te}^2/2 - \gamma_S \theta_{Se} = K(q_e).$$

Here K stands for a real differentiable function. Note that this becomes the Bernoulli condition for incompressible 2D Euler upon substitution of $\theta_S = 0$, $u_T = 0$. To provide an explicit formula for the function K , we express (4.27) as

$$q_e \hat{z} \times u_{Se} = -s \nabla K(q_e).$$

Applying cross product with \hat{z} on the left-hand side we have that

$$-q_e u_{Se} = -s \hat{z} \times \nabla K(q_e).$$

Therefore, by taking into account the equilibrium condition (1.42),

$$q_e \text{curl}(\Phi'(q_e) \hat{z}) = -\hat{z} \times \nabla K(q_e).$$

Rewrite this as

$$q_e \nabla^T(\Phi'(q_e)) = \nabla^T(K(q_e)),$$

and applying the chain rule

$$q_e \Phi''(q_e) \nabla^T q_e = K'(q_e) \nabla^T q_e,$$

so we infer that

$$q_e \Phi''(q_e) = K'(q_e).$$

Hence, upon integration, we find that the function Φ can be expressed as

$$\Phi(\lambda) = \lambda \left(\int_{\lambda} \frac{K(t)}{t^2} dt + C \right),$$

where C is an integration constant. Also, since $\nabla K(q) = K'(q) \nabla q$, we obtain

$$\Phi''(q_e) = \frac{(\hat{z} \times \nabla q_e) \cdot u_{Se}}{|\nabla q_e|^2}. \quad (4.28)$$

proving Theorem 1.4.

Remark 4.13. Relation (4.28) is fundamental when deriving formal and nonlinear stability conditions for the ISM.

4.3.2 Formal stability conditions for the ISM: Proof of Theorem 1.5

To study the formal stability of the ISM around our restricted class of equilibrium solutions, we need to calculate the second variation of H_Φ , which reads

$$\delta^2 H_\Phi = \int_{\Omega} \left\{ |\delta u_S|^2 + (\delta u_T)^2 \right\} dV + \int_{\Omega} \Phi''(q)(\delta q)^2 dV.$$

Therefore, one obtains a straightforward (albeit quite general) condition for formal stability, namely

$$\Phi''(q_e) > 0,$$

as state in Theorem 1.5.

4.3.3 Nonlinear stability for the ISM: Proof of Theorem 1.6

To derive nonlinear stability conditions for the ISM, we follow the Energy-Casimir algorithm (i)-(vi). We need to construct two quadratic forms Q_1 and Q_2 depending on the variables δu_S , $\delta \theta_S$, δu_T . Note that Q_1 has to satisfy

$$\begin{aligned} Q_1(\Delta u_S, \Delta u_T, \Delta \theta_S) &\leq h(u_{Se} + \Delta u_S, u_{Te} + \Delta u_T, \theta_{Se} + \Delta \theta_S) - h(u_{Se}, u_{Te}, \theta_{Se}) \\ &\quad - Dh(u_{Se}, u_{Te}, \theta_{Se}) \cdot (\Delta u_S, \Delta u_T, \Delta \theta_S). \end{aligned}$$

Since the conserved Hamiltonian h is quadratic plus a linear term on θ_S (cf. (4.20)), this can be done by choosing $Q_1 = h + \gamma_S \theta_S$. Next, since Q_2 has to satisfy

$$\begin{aligned} Q_2(\Delta u_S, \Delta u_T, \Delta \theta_S) &\leq C_\Phi(u_{Se} + \Delta u_S, u_{Te} + \Delta u_T, \theta_{Se} + \Delta \theta_S) - C_\Phi(u_{Se}, u_{Te}, \theta_{Se}) \\ &\quad - DC_\Phi(u_{Se}, u_{Te}, \theta_{Se}) \cdot (\Delta u_S, \Delta u_T, \Delta \theta_S), \end{aligned}$$

we select

$$Q_2 = \lambda_1 \int_{\Omega} (\Delta q)^2 dV,$$

where $\lambda_1 \in \mathbb{R}$ is such that

$$\lambda_1 \leq \Phi''(x), \quad \text{for all } x \in \mathbb{R}.$$

Condition (iv) in the Energy-Casimir algorithm requires

$$Q_1 + Q_2 = h + \lambda_1 \int_{\Omega} (\Delta q)^2 dV > 0,$$

which is guaranteed for instance if $\lambda_1 > 0$. We point out that H_Φ is continuous with respect to the defined norm

$$\|(\delta u_S, \delta u_T, \delta \theta_S)\|_{Q_1+Q_2} = \int_{\Omega} \left\{ \frac{1}{2} |\delta u_S|^2 + \frac{1}{2} (\delta u_T)^2 \right\} dV + \lambda_1 \int_{\Omega} (\delta q)^2 dV,$$

provided $\Phi''(x) \leq \lambda_2 < \infty$, for all $x \in \mathbb{R}$.

Finally, we show the estimate required in (v):

$$\begin{aligned} Q_1 + Q_2 &= \int_{\Omega} \left\{ |\Delta u_S(t)|^2 + (\Delta u_T(t))^2 \right\} dV + \lambda_1 \int_{\Omega} (\Delta q(t))^2 dV \\ &\leq \int_{\Omega} \left\{ \frac{1}{2} (|u_S(0)|^2 + u_T(0)^2) - \gamma_S \theta_S(0) \right\} dV + \int_{\Omega} \Phi(q(0)) dV \\ &\quad - \int_{\Omega} \left\{ \frac{1}{2} (|u_{Se}|^2 + u_{Te}^2) - \gamma_S \theta_{Se} \right\} dV - \int_{\Omega} \Phi(q_e) dV. \end{aligned}$$

Therefore, taking into account the construction above and Theorem 4.2 we have derived the proof of Theorem 1.6.

4.4 Local well-posedness of the ISM

In this section, we establish the local existence and uniqueness of solutions of (4.16a)-(4.16d) on bounded domains $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$ satisfying the boundary condition (4.17), stated in Theorem 1.7. We will assume that the constants $f = s = \theta_0 = g = 1$ without loss of generality. Therefore, the equations can be written as

$$\partial_t u_S + u_S \cdot \nabla u_S - u_T \hat{x} = -\nabla p + \theta_S \hat{z}, \quad (4.29a)$$

$$\partial_t u_T + u_S \cdot \nabla u_T + u_S \cdot \hat{x} = -z, \quad (4.29b)$$

$$\partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T = 0, \quad (4.29c)$$

$$\nabla \cdot u_S = 0, \quad (4.29d)$$

with boundary condition

$$u_S \cdot n = 0 \quad \text{on } \partial\Omega. \quad (4.30)$$

Before starting with the proof of Theorem 1.7, we project our equations (4.29a)-(4.29d) by using the Leray's projector \mathbb{P} (cf. Lemma 4.2) into the following new system of equations

$$\partial_t u_S + \mathbb{P}(u_S \cdot \nabla u_S) - \mathbb{P}(u_T \hat{x}) = \mathbb{P}(\theta_S \hat{z}), \quad (4.31a)$$

$$\partial_t u_T + u_S \cdot \nabla u_T + u_S \cdot \hat{x} = -z, \quad (4.31b)$$

$$\partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T = 0, \quad (4.31c)$$

$$\mathbb{P}u_S = u_S. \quad (4.31d)$$

We will show that we can go from (4.31a)-(4.31d) to (4.29a)-(4.29d) by solving a Poisson problem for the pressure. The following lemma is well-known [LM61],

Lemma 4.3. Given $f \in W^{k,p}(\Omega)$ for $k \in \mathbb{N}$, and $g \in W^{k+1-\frac{1}{p},p}(\partial\Omega)$ satisfying the compatibility condition

$$\int_{\Omega} f \, dV = \int_{\partial\Omega} g \, dV,$$

there exists $\phi \in W^{k+2,p}(\Omega)$ satisfying

$$\begin{aligned} \Delta\phi &= f, \quad \text{in } \Omega, \\ \frac{\partial\phi}{\partial n} &= g, \quad \text{on } \partial\Omega. \end{aligned} \tag{4.32}$$

Moreover,

$$\|\nabla\phi\|_{W^{k+1,p}(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)} + \|g\|_{W^{k+1-\frac{1}{p},p}(\partial\Omega)}.$$

Let us also denote $F(f, g) := f \cdot \nabla g$, where f, g can be vector fields on Ω or a scalar functions. In order to prove that (4.31a)-(4.31d) admits a unique local strong solution, we will apply Theorem 4.3 to the following variation of the aforementioned equations, in which the solution is sought in $H^s \times H^s \times H^s$ rather than in $H^s_\star \times H^s \times H^s$:

$$\partial_t u_S + F(\mathbb{P}u_S, u_S) - \mathbb{Q}F(\mathbb{P}u_S, \mathbb{P}u_S) - \mathbb{P}(u_T \hat{x}) - \mathbb{P}(\theta_S \hat{z}) = 0, \tag{4.33a}$$

$$\partial_t u_T + F(\mathbb{P}u_S, u_T) + \mathbb{P}u_S \cdot \hat{x} + z = 0, \tag{4.33b}$$

$$\partial_t \theta_S + F(\mathbb{P}u_S, \theta_S) + u_T = 0. \tag{4.33c}$$

We claim that the condition $\mathbb{P}u_S = u_S$ follows immediately for all the solutions $u_S \in C_w([0, T]; H^s)$ of equation (4.33a)-(4.33c). Indeed, first note that

$$(\mathbb{P}(u_T \hat{x} + \theta_S \hat{z}), \mathbb{Q}u_S)_{L^2} = 0, \tag{4.34}$$

$$(\mathbb{Q}F(\mathbb{P}u_S, \mathbb{P}u_S), \mathbb{Q}u_S)_{L^2} = (F(\mathbb{P}u_S, \mathbb{P}u_S), \mathbb{Q}u_S)_{L^2}. \tag{4.35}$$

Therefore, using (4.34) and (4.35) we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbb{Q}u_S\|_{L^2}^2 &= (\partial_t u_S, \mathbb{Q}u_S)_{L^2} \\ &= -(F(\mathbb{P}u_S, u_S - \mathbb{P}u_S), \mathbb{Q}u_S)_{L^2} = -(F(\mathbb{P}u_S, \mathbb{Q}u_S), \mathbb{Q}u_S)_{L^2} = 0, \end{aligned}$$

since $(F(f, g), g)_{L^2} = 0$ if f is divergence-free. Hence, any weak solution $u_S \in C_w([0, T]; H^s)$ of (4.33a)-(4.33c) satisfies $\mathbb{Q}u_S = 0$.

Now we state some estimates we will need in the proof of Theorem 1.7:

Proposition 4.1. Set $s_0 = 3$ and $s \in \mathbb{N}$.

(i) Let $s \geq s_0$.

a) Let $f \in H_\star^s$, $g \in H^{s+1}$. We have that

$$\|F(f, g)\|_{H^s} \lesssim \|f\|_{H^{s_0-1}} \|g\|_{H^{s+1}} + \|f\|_{H^s} \|g\|_{H^{s_0}}, \quad (4.36)$$

$$|(g, F(f, g))_{H^s}| \lesssim \|f\|_{H^{s_0}} \|g\|_{H^s}^2 + \|f\|_{H^s} \|g\|_{H^s} \|g\|_{H^{s_0}}. \quad (4.37)$$

b) Let $f, g \in H_\star^s$. Then $\mathbb{Q}SF(f, g) \in H^s$ and

$$\|\mathbb{Q}F(f, g)\|_{H^s} \lesssim \|f\|_{H^{s_0}} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{H^{s_0}}.$$

c) Let $f \in H_\star^s$, $g \in H_\star^{s+1}$. We have

$$|(g, \mathbb{P}F(f, g))_{H^s}| \lesssim \|f\|_{H^{s_0}} \|g\|_{H^s}^2 + \|f\|_{H^s} \|g\|_{H^s} \|g\|_{H^{s_0}}.$$

(ii) Let $1 \leq s \leq s_0 - 1$.

a) For $f \in H_\star^{s_0}$, $g \in H^{s+1}$

$$\|F(f, g)\|_{H^s} \lesssim \|f\|_{H^{s_0-1}} \|g\|_{H^{s+1}}, \quad (4.38)$$

$$|(g, F(f, g))_{H^s}| \lesssim \|f\|_{H^{s_0}} \|g\|_{H^s}^2. \quad (4.39)$$

b) If $f \in H_\star^{s_0}$, $g \in H_\star^s$ we have

$$\|\mathbb{Q}(f, g)\|_{H^s} \lesssim \|f\|_{H^{s_0}} \|g\|_{H^s}.$$

c) If $f \in H_\star^{s_0}$, $g \in H_\star^{s+1}$ then

$$|(f, \mathbb{P}F(f, g))_{H^s}| \lesssim \|f\|_{H^{s_0}} \|g\|_{H^s}^2.$$

Proof of Proposition 4.1. All the estimates are quite standard and are based on well known calculus inequalities, cf. [KL84; Kat72] \square

4.4.1 Proof of Theorem 1.7

We are ready to start with the proof of Theorem 1.7. We wish to apply Theorem 4.3 to equations (4.33a)-(4.33c). To do this we need to construct an admissible triplet $\{V, H, X\}$. Define $X = H^0 \times H^0 \times H^0$, and for $s \geq s_0$ set $H = H^{s, s_0}$ to be the Hilbert space equipped with the norm

$$\begin{aligned} & ((u_S, u_T, \theta_S), (u'_S, u'_T, \theta'_S))_H \\ &= (u_S, u'_S)_{H^{s_0}} + (u_S, u'_S)_{H^s} + (u_T, u'_T)_{H^{s_0}} + (u_T, u'_T)_{H^s} + (\theta_S, \theta'_S)_{H^{s_0}} + (\theta_S, \theta'_S)_{H^s}. \end{aligned}$$

Note that since $s \geq s_0$, the H -norm is equivalent to the H^s -norm because of the inequalities

$$\|f\|_{H^s}^2 \leq \|f\|_H^2 \leq 2\|f\|_{H^s}^2, \quad f \in H.$$

We are left to construct V . This is selected as the subspace of H^s corresponding to the domain of the self-adjoint unbounded nonnegative operator S taking values in H^0 defined by

$$S = \sum_{|\alpha| \leq s_0} (-1)^\alpha D^{2\alpha} + \sum_{|\alpha| \leq s} (-1)^\alpha D^{2\alpha},$$

with Neumann boundary conditions. Then one has (cf. [Nir55; Lio69])

$$V \subset H^{2s} \subset H^{s+1}.$$

Note that we have not used H_*^m or H_*^0 to define the admissible triplet above. This is due to the extra complications that arise from the divergence and boundary conditions in order to define a suitable space V .

We apply Theorem 4.3 to the operator

$$A(t, (u_S, u_T, \theta_S)) = (A_1(t, (u_S, u_T, \theta_S)), A_2(t, (u_S, u_T, \theta_S)), A_3(t, (u_S, u_T, \theta_S))),$$

where

$$\begin{aligned} A_1 &= F(\mathbb{P}u_S, u_S) - \mathbb{Q}F(\mathbb{P}u_S, \mathbb{P}u_S) - \mathbb{P}(u_T \hat{x}) - \mathbb{P}(\theta_S \hat{z}), \\ A_2 &= F(\mathbb{P}u_S, u_T) + \mathbb{P}u_S \cdot \hat{x} - z, \\ A_3 &= F(\mathbb{P}u_S, \theta_S) + u_T. \end{aligned}$$

Due to Proposition 4.1 ((i)a-(i)b and (ii)a-(ii)b), the operator A maps $[0, T] \times H = [0, T] \times H^s \times H^s \times H^s$ into $H^{s-1} \times H^{s-1} \times H^{s-1} \subset H^0 = X$ weakly continuously. Now we bound

$$\begin{aligned} ((u_S, u_T, \theta_S), A(t, (u_S, u_T, \theta_S)))_H &= (u_S, F(\mathbb{P}u_S, u_S) - \mathbb{Q}F(\mathbb{P}u_S, \mathbb{P}u_S) - \mathbb{P}(u_T \hat{x}) - \mathbb{P}(\theta_S \hat{z}))_{H^{s_0}} \\ &\quad + (u_S, F(\mathbb{P}u_S, u_S) - \mathbb{Q}F(\mathbb{P}u_S, \mathbb{P}u_S) - \mathbb{P}(u_T \hat{x}) - \mathbb{P}(\theta_S \hat{z}))_{H^s} \\ &\quad + (u_T, F(\mathbb{P}u_S, u_T) + \mathbb{P}u_S \cdot \hat{x} - z)_{H^{s_0}} \\ &\quad + (u_T, F(\mathbb{P}u_S, u_T) + \mathbb{P}u_S \cdot \hat{x} - z)_{H^s} \\ &\quad + (\theta_S, F(\mathbb{P}u_S, \theta_S) + u_T)_{H^{s_0}} + (\theta_S, F(\mathbb{P}u_S, \theta_S) + u_T)_{H^s}. \end{aligned}$$

By using (i)a in Proposition 4.1, we obtain the estimates

$$\begin{aligned} |(u_S, F(\mathbb{P}u_S, u_S))_{H^{s_0}}| &\lesssim \|u_S\|_{H^{s_0}}^3, \\ |(u_T, F(\mathbb{P}u_S, u_T))_{H^{s_0}}| &\lesssim \|u_S\|_{H^{s_0}} \|u_T\|_{H^{s_0}}^2, \\ |(\theta_S, F(\mathbb{P}u_S, \theta_S))_{H^{s_0}}| &\lesssim \|u_S\|_{H^{s_0}} \|\theta_S\|_{H^{s_0}}^2. \end{aligned}$$

Moreover, by (i)a

$$|(u_S, \mathbb{Q}F(\mathbb{P}u_S, \mathbb{P}u_S))_{H^{s_0}}| \lesssim \|u_S\|_{H^{s_0}} \|\mathbb{Q}F(\mathbb{P}u_S, \mathbb{P}u_S)\|_{H^{s_0}} \lesssim \|u_S\|_{H^{s_0}}^3.$$

In the same way we can derive the bounds:

$$\begin{aligned} |(u_S, F(\mathbb{P}u_S, u_S))_{H^s}| &\lesssim \|u_S\|_{H^{s_0}} \|u_S\|_{H^s}^2, \\ |(u_T, F(\mathbb{P}u_S, u_T))_{H^s}| &\lesssim \|u_S\|_{H^{s_0}} \|u_T\|_{H^s}^2 + \|u_S\|_{H^s} \|u_T\|_{H^s} \|u_T\|_{H^{s_0}}, \\ |(\theta_S, F(\mathbb{P}u_S, \theta_S))_{H^s}| &\lesssim \|u_S\|_{H^{s_0}} \|\theta_S\|_{H^s}^2 + \|u_S\|_{H^s} \|\theta_S\|_{H^s} \|\theta_S\|_{H^{s_0}}. \end{aligned}$$

Also, by (i)a

$$|(u_S, \mathbb{Q}F(\mathbb{P}u_S, \mathbb{P}u_S))_{H^s}| \lesssim \|u_S\|_{H^s} \|\mathbb{Q}F(\mathbb{P}u_S, \mathbb{P}u_S)\|_{H^s} \lesssim \|u_S\|_{H^s}^2 \|u_S\|_{H^{s_0}}.$$

Note that the linear terms $\mathbb{P}(u_T \hat{x}), \mathbb{P}(\theta_S \hat{z}), u_S \cdot \hat{x}, z, u_T$, can be straightforwardly estimated. Therefore

$$\begin{aligned} ((u_S, u_T, \theta_S), A(t, (u_S, u_T, \theta_S)))_H &\lesssim (\|u_S\|_{H^{s_0}} + \|u_T\|_{H^{s_0}} + \|\theta_S\|_{H^{s_0}})^3 \\ &\quad + (\|u_S\|_{H^{s_0}} + \|u_T\|_{H^{s_0}} + \|\theta_S\|_{H^{s_0}} + 1) \\ &\quad \times (\|u_S\|_{H^s} + \|u_T\|_{H^s} + \|\theta_S\|_{H^s})^2 \\ &\lesssim \|(u_S, u_T, \theta_S)\|_H^3 + \|(u_S, u_T, \theta_S)\|_H^2 + \|(u_S, u_T, \theta_S)\|_H \\ &\lesssim (\|(u_S, u_T, \theta_S)\|_H + 1)^3. \end{aligned} \quad (4.40)$$

Hence, Theorem 4.3 can be applied to equations (4.33a)-(4.33c) with initial data $(u_S^0, u_T^0, \theta_S^0)$ in H^s , guaranteeing the existence of a locally weak solution $(u_S, u_T, \theta_S) \in C_w([0, T]; H^s \times H^s \times H^s)$ for any integer $s > 2$.

Estimate (4.40) is not sharp. Obtaining better estimates for the existence time is just a matter of carrying out the estimates more carefully. However, for the sake of exposition clarity it was convenient to write them as we did. The bound condition in (4.3) is satisfied with

$$\beta(r) = \frac{K}{2}(r+1)^{3/2},$$

so the differential equation (4.4) (to be solved in order to obtain a local existence time) becomes

$$\begin{cases} \rho_t = K(\rho+1)^{3/2}, \\ \rho(0) = \|\phi\|_H^2. \end{cases} \quad (4.41)$$

The solution of this ODE can be explicitly written as

$$\rho(t) = \frac{-2CKt - C^2 - (2Kt)^2 + 4}{(C + Kt)^2}, \quad \text{with} \quad C = \frac{2}{\sqrt{\|\phi\|_H^2 + 1}}.$$

This means the solution of (4.41) is at least guaranteed to exist until

$$T = \frac{2}{K\sqrt{\|\phi\|_H^2 + 1}}.$$

To complete the proof of Theorem 1.7, we need to justify the regularity and uniqueness of the solutions. First, we focus on showing uniqueness of the weak solution. Indeed, first note that $(u_S, u_T, \theta_S) \in \text{Lip}([0, T]; H_\star^0 \times H^0 \times H^0)$ since $\partial_t(u_S, u_T, \theta_S) \in C_w([0, T]; H_\star^0 \times H^0 \times H^0)$, so $\|(u_S, u_T, \theta_S)\|_{L^2}$ is differentiable in time. To show uniqueness we argue by contradiction. Suppose that $(u_S^1, u_T^1, \theta_S^1)$ and $(u_S^2, u_T^2, \theta_S^2)$ are two different solutions of (4.33a)-(4.33c) with the same initial data $(u_S^0, u_T^0, \theta_S^0)$. Define the differences $\tilde{u}_S = u_S^1 - u_S^2$, $\tilde{u}_T = u_T^1 - u_T^2$, $\tilde{\theta}_S = \theta_S^1 - \theta_S^2$. We have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{u}_S\|_{L^2}^2 &= -(\mathbb{P}(u_S^1 \cdot \nabla \tilde{u}_S), \tilde{u}_S)_{L^2} - (\mathbb{P}(\tilde{u}_S \cdot \nabla u_S^2), \tilde{u}_S)_{L^2} + (\mathbb{P}(\tilde{u}_T \hat{x}), \tilde{u}_S)_{L^2} + (\mathbb{P}(\tilde{\theta}_S \hat{z}), \tilde{u}_S)_{L^2}, \\ \frac{1}{2} \frac{d}{dt} \|\tilde{u}_T\|_{L^2}^2 &= -(\mathbb{P}u_S^1 \cdot \nabla \tilde{u}_T, \tilde{u}_T)_{L^2} - (\mathbb{P}\tilde{u}_S \cdot \nabla u_T^2, \tilde{u}_T)_{L^2} - (\mathbb{P}\tilde{u}_S \cdot \hat{x}, \tilde{u}_T)_{L^2}, \\ \frac{1}{2} \frac{d}{dt} \|\tilde{\theta}_S\|_{L^2}^2 &= -(\mathbb{P}u_S^1 \cdot \nabla \tilde{\theta}_S, \tilde{\theta}_S)_{L^2} - (\mathbb{P}\tilde{u}_S \cdot \nabla \theta_S^2, \tilde{\theta}_S)_{L^2} - (\tilde{u}_T, \tilde{\theta}_S)_{L^2}. \end{aligned}$$

By using the properties of \mathbb{P} stated in Lemma 4.2 we observe that

$$(\mathbb{P}(u_S^1 \cdot \nabla \tilde{u}_S), \tilde{u}_S)_{L^2} = (u_S^1 \cdot \nabla \tilde{u}_S, \mathbb{P}\tilde{u}_S)_{L^2} = (u_S^1 \cdot \nabla \tilde{u}_S, \tilde{u}_S)_{L^2} = 0,$$

and

$$(\mathbb{P}u_S^1 \cdot \nabla \tilde{u}_T, \tilde{u}_T)_{L^2} = 0, \quad (\mathbb{P}u_S^1 \cdot \nabla \tilde{\theta}_S, \tilde{\theta}_S)_{L^2} = 0.$$

Hence obtaining

$$\frac{d}{dt} \left(\|\tilde{u}_S\|_{L^2}^2 + \|\tilde{u}_T\|_{L^2}^2 + \|\tilde{\theta}_S\|_{L^2}^2 \right) \lesssim A(t) \left(\|\tilde{u}_S\|_{L^2}^2 + \|\tilde{u}_T\|_{L^2}^2 + \|\tilde{\theta}_S\|_{L^2}^2 \right),$$

with $A(t) = (\|\nabla u_S^2\|_{L^\infty} + \|\nabla u_T^2\|_{L^\infty} + \|\nabla \theta_S^2\|_{L^\infty} + 1)$, from which, after an application of Grönwall's argument, uniqueness follows.

We have shown we can construct a unique weak local solution of (4.33a)-(4.33c). Moreover, the solution actually enjoys higher regularity. This is the last point we need to discuss to conclude our argument. We demonstrate this in four steps:

Step 1: First note that although the solution $(u_S(t), u_T(t), \theta_S(t))$ of (4.33a)-(4.33c) was sought in $C_w([0, T]; H^s \times H^s \times H^s)$ with initial data ϕ , we actually have $\|u(t)\|_H \rightarrow \|\phi\|_H$ (strongly!). This holds because $\rho(t) \rightarrow \|\phi\|_H^2$ for the solution of the differential equation (4.4).

Step 2: By Step 1, (u_S, u_T, θ_S) is right-continuous at $T = 0$. It is easy to check

that it must also be right-continuous on $[0, T]$ by uniqueness (for $t \in [0, T]$ consider the same PDE with initial data $(u_S, u_T, \theta_S)(t, \cdot)$).

Step 3: We need to prove that (u_S, u_T, θ_S) is also left-continuous. However, since the ISM system (4.29a)-(4.29d) is not time reversible, the customary argument does not apply. We make the following change of variables:

$$\tilde{u}_S(X, t) := u_S(-X, -t), \quad \tilde{u}_T(X, t) := -u_T(-X, -t), \quad \tilde{\theta}_S(X, t) := -\theta_S(-X, -t),$$

with $X = (x, z)$. The new functions satisfy the following system of equations

$$\partial_t \tilde{u}_S + \mathbb{P}(\tilde{u}_S \cdot \nabla \tilde{u}_S) - \mathbb{P}(\tilde{u}_T \hat{x}) = \mathbb{P}(\tilde{\theta}_S \hat{z}), \quad (4.42a)$$

$$\partial_t \tilde{u}_T + \tilde{u}_S \cdot \nabla \tilde{u}_T + \tilde{u}_S \cdot \hat{x} = -z, \quad (4.42b)$$

$$\partial_t \tilde{\theta}_S + \tilde{u}_S \cdot \nabla \tilde{\theta}_S - \tilde{u}_T = 0, \quad (4.42c)$$

$$\mathbb{P} \tilde{u}_S = \tilde{u}_S. \quad (4.42d)$$

The operator \tilde{A} associated with (4.42a)-(4.42d) satisfies estimates which are similar to the ones satisfied by A (4.33a)-(4.33c). Therefore we can apply the same weak existence and uniqueness theorem in $C_w([0, \tilde{T}]; H^s \times H^s \times H^s)$ with initial data in H^s for our new variables $(\tilde{u}_S(X, t), \tilde{u}_T(X, t), \tilde{\theta}_S(X, t))$. The existence time \tilde{T} of the new tilde solutions needs not be the same as the one for the standard ones. Also, by Step 2, the new tilde solutions of (4.42a)-(4.42d) are necessarily right-continuous.

Step 4: Now let $(u_S, u_T, \theta_S) \in C_w([0, T]; H^s \times H^s \times H^s)$ solve (4.33a)-(4.33c). We need to prove that (u_S, u_T, θ_S) is left-continuous. For this, construct

$$(\tilde{u}_S(t), \tilde{u}_T(t), \tilde{\theta}_S(t)) = (u_S(T - t, \cdot), u_T(T - t, \cdot), \theta_S(T - t, \cdot)) \in C_w([0, T]; H^s \times H^s \times H^s),$$

with $\tilde{\phi} = (\tilde{u}_S(0, \cdot), \tilde{u}_T(0, \cdot), \tilde{\theta}_S(0, \cdot)) \in H^s \times H^s \times H^s$. Therefore $(\tilde{u}_S, \tilde{u}_T, \tilde{\theta}_S)$ has to be the unique weak local solution of (4.42a)-(4.42d) with initial data $\tilde{\phi}$. By Step 3, $(\tilde{u}_S, \tilde{u}_T, \tilde{\theta}_S)$ is right-continuous, and hence (u_S, u_T, θ_S) is left-continuous.

We have shown that there exists a local solution of (4.33a)-(4.33c) in $C([0, T]; H^s \times H^s \times H^s)$, concluding the proof.

For the sake of completeness we also include, without proof, the following result which states the continuous dependence of the solution on the initial data:

Theorem 4.6. *Let $s > 2$ be an integer. Let $(u_S, u_T, \theta_S) \in C([0, T]; H^s_\star \times H^s \times H^s)$ be the solution of (4.29a)-(4.29d) with initial data $(u_S^0, u_T^0, \theta_S^0) \in H^s_\star \times H^s \times H^s$. Let $(u_S^j, u_T^j, \theta_S^j)$ be the solution of (4.29a)-(4.29d) with initial data $(u_S^{0,j}, u_T^{0,j}, \theta_S^{0,j}) \in$*

$H^s \times H^s \times H^s$ $j = 1, 2, \dots$, such that $(u_S^{0,j}, u_T^{0,j}, \theta_S^{0,j}) \rightarrow (u_S^0, u_T^0, \theta_S^0)$ in $H^s \times H^s \times H^s$. Then

$$(u_S^j, u_T^j, \theta_S^j) \rightarrow (u_S, u_T, \theta_S), \quad \text{in } C([0, T]; H^s \times H^s \times H^s).$$

Remark 4.14. We do not provide the proof of Theorem 4.6 since it can be performed by imitating the arguments in [KL84].

4.5 Blow-up criterion for the ISM

In this section we provide the proof of the blow-up criterion for the Incompressible Slice Model stated in Theorem 1.8. Before starting with the proof itself, let us make some remarks about the blow-up criteria.

Remark 4.15. Theorem 1.8 can be a very useful tool for studying the problem of blow-up versus global existence in the Incompressible Slice Model. Note that, for instance, it can be applied to check whether a numerical simulation shows blow-up in finite time.

Remark 4.16. Notice that we cannot expect (as one has in 3D Euler), a Beale-Kato-Majda criterion, stating that if

$$\int_0^{T^*} \|\omega\|_\infty < \infty,$$

then the corresponding solution stays regular on $[0, T^*]$. Here $\omega = \text{curl } u$. The main problem is that we cannot control properly θ_S and u_T in terms of the vorticity only. If the equations were not coupled, this could be easily done by using a logarithmic inequality as in 3D Euler. However, in the ISM case it seems that controlling $\|\omega\|_\infty$ is not enough for global regularity.

4.5.1 Proof of Theorem 1.8

The proof is divided into several steps. *Step 1: Estimates for $\|u_S\|_{H^s}$, $\|u_T\|_{H^s}$, and $\|\theta_S\|_{H^s}$.* Setting $|\alpha| \leq s$, we apply D^α on both sides of (4.29a), multiply by $D^\alpha u_S$ and integrate over Ω to obtain

$$(D^\alpha \partial_t u_S, D^\alpha u_S)_{L^2} = -(D^\alpha (u_S \cdot \nabla u_S), D^\alpha u_S)_{L^2} + (D^\alpha (u_T \hat{x}), D^\alpha u_S)_{L^2} \quad (4.43)$$

$$+ (D^\alpha (\theta_S \hat{z}), D^\alpha u_S)_{L^2} - (D^\alpha \nabla p, D^\alpha u_S)_{L^2}. \quad (4.44)$$

The first term on the right-hand side of (4.43) can be rewritten as

$$(D^\alpha (u_S \cdot \nabla u_S), D^\alpha u_S)_{L^2} = (D^\alpha (u_S \cdot \nabla u_S) - u_S \cdot D^\alpha \nabla u_S, D^\alpha u_S)_{L^2} + (u_S \cdot D^\alpha \nabla u_S, D^\alpha u_S)_{L^2}. \quad (4.45)$$

Therefore, applying the Cauchy-Schwarz inequality in (4.43)-(4.44), we have that

$$\begin{aligned} \frac{d}{dt} \|D^\alpha u_S\|_{L^2}^2 &\lesssim \|D^\alpha u_S\|_{L^2} \|D^\alpha (u_S \cdot \nabla u_S) - u_S \cdot D^\alpha \nabla u_S\|_{L^2} \\ &\quad + \|D^\alpha u_S\|_{L^2} (\|D^\alpha (u_T \hat{x})\|_{L^2} + \|D^\alpha (\theta_S \hat{z})\|_{L^2} + \|D^\alpha \nabla p\|_{L^2}), \end{aligned}$$

where we have used that the last term in (4.45) vanishes after integration by parts due to $\operatorname{div} u_S = 0$ and $u_S \cdot n = 0$ on $\partial\Omega$. Summing over $|\alpha| \leq s$, applying Young's inequality, and using (ii) in Lemma 4.1 (with $f = u_S$ and $g = \nabla u_S$) yields

$$\frac{d}{dt} \|u_S\|_{H^s}^2 \lesssim \|u_S\|_{H^s}^2 (\|u_S\|_{W^{1,\infty}} + 1) + \|u_T\|_{H^s}^2 + \|\theta_S\|_{H^s}^2 + \|u_S\|_{H^s} \|\nabla p\|_{H^s}. \quad (4.46)$$

To estimate $\|u_T\|_{H^s}$ and $\|\theta_S\|_{H^s}$ we proceed in a similar fashion. By applying these same techniques to (4.29b), using Lemma 4.1, and Cauchy-Schwarz inequality, we derive

$$\frac{d}{dt} \|D^\alpha u_T\|_{L^2}^2 \lesssim \|D^\alpha u_T\|_{L^2} (\|u_S\|_{W^{1,\infty}} \|u_T\|_{H^s} + \|u_S\|_{H^s} \|\nabla u_T\|_{L^\infty} + \|D^\alpha u_S\|_{L^2} + \|D^\alpha z\|_{L^2}).$$

Summing over $|\alpha| \leq s$ and using Young's inequality gives

$$\frac{d}{dt} \|u_T\|_{H^s}^2 \lesssim \|u_T\|_{H^s}^2 \|u_S\|_{W^{1,\infty}} + \|\nabla u_T\|_{L^\infty} \|u_S\|_{H^s} \|u_T\|_{H^s} + \|u_S\|_{H^s}^2 + \|u_T\|_{H^s}^2 + 1. \quad (4.47)$$

A similar argument can be carried out to estimate $\|\theta_S\|_{H^s}$, obtaining

$$\frac{d}{dt} \|\theta_S\|_{H^s}^2 \lesssim \|\theta_S\|_{H^s}^2 \|u_S\|_{W^{1,\infty}} + \|\nabla \theta_S\|_{L^\infty} \|u_S\|_{H^s} \|\theta_S\|_{H^s} + \|\theta_S\|_{H^s}^2 + \|u_T\|_{H^s}^2. \quad (4.48)$$

Finally, by (4.46), (4.47), and (4.48), we conclude

$$\frac{d}{dt} E(t) \lesssim (\|u_S\|_{W^{1,\infty}} + \|\nabla u_T\|_{L^\infty} + \|\nabla \theta_S\|_{L^\infty} + 1) (E(t) + 1) + \|\nabla p\|_{H^s} \|u_S\|_{H^s}, \quad (4.49)$$

where $E(t) = \|u_S\|_{H^s}^2 + \|u_T\|_{H^s}^2 + \|\theta_S\|_{H^s}^2$.

Step 2: Estimate the pressure term $\|\nabla p\|_{H^s}$. In order to do this, we need the following lemma.

Lemma 4.4. *If u_S, u_T, θ_S , and p satisfy equations (4.29a)-(4.29d) with boundary condition (4.30), then for $s \geq 3$ we have the following estimate*

$$\|\nabla p\|_{H^s} \lesssim \|u_S\|_{H^s} \|u_S\|_{W^{1,\infty}} + \|u_T\|_{H^s} + \|\theta_S\|_{H^s}.$$

Proof of Lemma 4.4. The proof follows closely the lines of [Tem75] for the incompressible Euler equation. There, $\|\nabla p\|_{H^s}$ is bounded in terms of $\|u\|_{H^s}^2$. We perform some simple modifications of this idea in order to obtain convenient estimates. First take the divergence in (4.29a) and dot it against the outward normal $n = (n_1, n_2)$.

Using the divergence-free condition and $u_S \cdot n = 0$ on $\partial\Omega$, we obtain the following Neumann problem for p

$$\Delta p = \operatorname{div} (u_T \hat{x} + \theta_S \hat{z}) - \sum_{i,j=1}^2 D_j u_{S,i} \cdot D_i u_{S,j}, \text{ in } \Omega, \quad (4.50)$$

$$\frac{\partial p}{\partial n} = (u_T \hat{x} + \theta_S \hat{z}) \cdot n - \sum_{i,j=1}^2 u_{S,i} (D_i u_{S,j}) n_j, \text{ on } \partial\Omega. \quad (4.51)$$

Proceeding as in [Tem75], we can eliminate the derivatives from u_S on the right hand side of (4.51) by representing $\partial\Omega$ locally as a level set of a smooth function, i.e. as $\phi(x) = 0$, so that on every local patch we can write

$$\frac{\partial p}{\partial n} = (u_T \hat{x} + \theta_S \hat{z}) \cdot n - \sum_{i,j=1}^2 u_{S,i} u_{S,j} \psi_{ij}, \text{ on } \partial\Omega,$$

where

$$\psi_{ij} = \frac{D_{ij} \phi(x)}{|\nabla \phi(x)|}.$$

Notice that this sort of representation is only possible because the boundary $\partial\Omega$ is smooth enough. Hence we can estimate the pressure term $\|\nabla p\|_{H^s}$ by applying (i) in Lemma 4.1, combined with the Trace Theorem 4.1. Indeed, by (4.50), (4.51), and Lemma 4.3:

$$\begin{aligned} \|\nabla p\|_{H^s} &\lesssim \|\operatorname{div}(u_T \hat{x} + \theta_S \hat{z}) - \sum_{i,j=1}^2 D_j u_{S,i} \cdot D_i u_{S,j}\|_{H^{s-1}(\Omega)} \\ &\quad + \|u_T \hat{x} + \theta_S \hat{z} \cdot n - \sum_{i,j=1}^2 u_{S,i} u_{S,j} \psi_{ij}\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \\ &\lesssim \|u_T \hat{x} + \theta_S \hat{z}\|_{H^s(\Omega)} + \left\| \sum_{i,j=1}^2 D_j u_{S,i} \cdot D_i u_{S,j} \right\|_{H^{s-1}(\Omega)} + \left\| \sum_{i,j=1}^2 u_{S,i} u_{S,j} \psi_{ij} \right\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \\ &\lesssim \|u_T \hat{x} + \theta_S \hat{z}\|_{H^s(\Omega)} \\ &\quad + \sum_{i,j=1}^2 \left(\|D_j u_{S,i}\|_{H^{s-1}(\Omega)} \|D_i u_{S,j}\|_{L^\infty(\Omega)} + \|D_j u_{S,i}\|_{L^\infty(\Omega)} \|D_i u_{S,j}\|_{H^{s-1}(\Omega)} \right) \\ &\quad + \sum_{i,j=1}^2 \|u_{S,i}\|_{H^s(\Omega)} \|u_{S,j}\|_{L^\infty(\Omega)} + \|u_{S,j}\|_{H^s(\Omega)} \|u_{S,i}\|_{L^\infty(\Omega)} \\ &\lesssim \|u_S\|_{H^s(\Omega)} \|u_S\|_{W^{1,\infty}(\Omega)} + \|u_T\|_{H^s(\Omega)} + \|\theta_S\|_{H^s(\Omega)}. \end{aligned}$$

□

Step 3: Controlling $\|\nabla u_T\|_{L^\infty}$ and $\|\nabla \theta_S\|_{L^\infty}$ by $\|\nabla u_S\|_{L^\infty}$. Take ∇ in (4.29b) to obtain

$$\partial_t \nabla u_T + \nabla(u_S \cdot \nabla u_T) + \nabla(u_S \cdot \hat{x}) = -\nabla z.$$

Let $p > 2$ be an integer and compute the L^2 inner product against $\nabla u_T |\nabla u_T|^{p-2}$ in the last equation, deriving

$$\begin{aligned} & (\partial_t \nabla u_T, \nabla u_T |\nabla u_T|^{p-2})_{L^2} + (\nabla(u_S \cdot \nabla u_T), \nabla u_T |\nabla u_T|^{p-2})_{L^2} \\ & + (\nabla(u_S \cdot \hat{x}), \nabla u_T |\nabla u_T|^{p-2})_{L^2} = (-\nabla z, \nabla u_T |\nabla u_T|^{p-2})_{L^2}. \end{aligned}$$

The first term on the left-hand side is

$$(\partial_t \nabla u_T, \nabla u_T |\nabla u_T|^{p-2})_{L^2} = \frac{1}{p} \frac{d}{dt} \|\nabla u_T\|_{L^p}^p.$$

We rewrite the second term as

$$(\nabla(u_S \cdot \nabla u_T), \nabla u_T |\nabla u_T|^{p-2})_{L^2} = ((\nabla u_S \cdot \nabla) u_T, \nabla u_T |\nabla u_T|^{p-2})_{L^2} + \frac{1}{p} (u_S, \nabla(|\nabla u_T|^p))_{L^2}. \quad (4.52)$$

Therefore, we obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\nabla u_T\|_{L^p}^p &= -((\nabla u_S \cdot \nabla) u_T, \nabla u_T |\nabla u_T|^{p-2})_{L^2} - (\nabla(u_S \cdot \hat{x}), \nabla u_T |\nabla u_T|^{p-2})_{L^2} \\ &\quad - (\nabla z, \nabla u_T |\nabla u_T|^{p-2})_{L^2}, \end{aligned}$$

where we have taken into account that the second term in the right-hand side of (4.52) vanishes after integration by parts. Now using Hölder and Young's inequality we derive

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\nabla u_T\|_{L^p}^p &\lesssim \|\nabla u_S\|_{L^\infty} \|\nabla u_T\|_{L^p}^p + \|\nabla u_S\|_{L^\infty} \|\nabla u_T\|_{L^p}^{p-1} + \|\nabla u_T\|_{L^p}^{p-1} \\ &\lesssim (\|\nabla u_S\|_{L^\infty} + 1) (\|\nabla u_T\|_{L^p}^p + 1). \end{aligned} \quad (4.53)$$

Proceeding as before, we can obtain a similar estimate for $\|\nabla \theta_S\|_{L^p}$ in the form

$$\frac{1}{p} \frac{d}{dt} \|\nabla \theta_S\|_{L^p}^p \lesssim \|\nabla u_S\|_{L^\infty} \|\nabla \theta_S\|_{L^p}^p + \frac{1}{p} \|\nabla \theta_S\|_{L^p}^{p-1} + \frac{1}{p} \|\nabla u_T\|_{L^p}^p. \quad (4.54)$$

Putting together (4.53) and (4.54) we conclude

$$\frac{d}{dt} (\|\nabla \theta_S\|_{L^p}^p + \|\nabla u_T\|_{L^p}^p) \lesssim p (\|\nabla u_S\|_{L^\infty} + 1) (\|\nabla \theta_S\|_{L^p}^p + \|\nabla u_T\|_{L^p}^p + 1).$$

It is important to note that the constant appearing implicitly in the above inequality does not depend on p . Thus, by applying Grönwall's inequality

$$\|\nabla \theta_S\|_{L^p}^p + \|\nabla u_T\|_{L^p}^p \lesssim \left(\|\nabla \theta_S^0\|_{L^p}^p + \|\nabla u_T^0\|_{L^p}^p \right) \exp \left(p \int_0^t (\|\nabla u_S(\tau)\|_{L^\infty} + 1) d\tau \right),$$

which gives

$$\|\nabla\theta_S\|_{L^p} + \|\nabla u_T\|_{L^p} \lesssim \left(\|\nabla\theta_S^0\|_{L^p} + \|\nabla u_T^0\|_{L^p} \right) \exp \left(\int_0^t (\|\nabla u_S(\tau)\|_{L^\infty} + 1) d\tau \right). \quad (4.55)$$

Finally, by taking limits in (4.55), we get

$$\begin{aligned} \|\nabla\theta_S\|_{L^\infty} + \|\nabla u_T\|_{L^\infty} &= \lim_{p \rightarrow \infty} (\|\nabla\theta_S\|_{L^p} + \|\nabla u_T\|_{L^p}) \\ &\lesssim \left(\|\nabla\theta_S^0\|_{L^\infty} + \|\nabla u_T^0\|_{L^\infty} \right) \exp \left(\int_0^t (\|\nabla u_S(\tau)\|_{L^\infty} + 1) d\tau \right). \end{aligned} \quad (4.56)$$

Step 4: Final stage and blow-up criterion. To conclude the proof we just need to collect estimates (4.49) and (4.56) to notice that setting $E(t) = 1 + \|u_S\|_{H^s}^2 + \|u_T\|_{H^s}^2 + \|\theta_S\|_{H^s}^2$, we have that

$$\dot{E}(t) \lesssim (\|\nabla\theta_S^0\|_{L^\infty} + \|\nabla u_T^0\|_{L^\infty}) \exp \left(\int_0^t (\|\nabla u_S(\tau)\|_{L^\infty} + 1) d\tau \right) E(t) + (\|u_S\|_{W^{1,\infty}} + 1) E(t).$$

By the Sobolev embedding there exists a constant $C > 0$ such that $\|\nabla u_S\|_{L^\infty} \leq CE(t)$. Consequently, using this fact and Grönwall's inequality, we obtain the equivalence between (1.46) and (1.47).

Remark 4.17. *We have derived the blow-up criterion theorem implicitly assuming that $u_S, u_T, \theta_S \in C([0, T]; H^{s+1})$, although it is only guaranteed that $u_S, u_T, \theta_S \in C([0, T]; H^s)$ by Theorem 1.8. The estimates can be made rigorous via standard approximation procedure and a routinary convergence argument. We shall omit this part to avoid redundancy.*

Global weak solutions

In this appendix we provide the existence theory of global weak solutions to incompressible drift diffusion equations on compact orientable manifolds given by

$$\begin{cases} \partial_t \theta + u \cdot \nabla_g \theta + \Lambda_g \theta = 0, \\ \nabla \cdot u = 0, \end{cases}$$

where the velocity field $u = \Psi[\theta]$, with Ψ a zero order pseudodifferential operator. This is a rather more general type of equations which has as a particular example the critical SQG (2.1) (for $\Psi = R_g^\perp$ and $M = \mathbb{S}^2$). More precisely, we prove the next result:

Theorem A.1. *Let $\theta_0 \in L^2(M)$ be the initial data and let $T > 0$. Then there exists a weak solution $\theta \in L^\infty(0, T; L^2(M)) \cap L^2(0, T; H^{1/2}(M))$ to the equation above i.e., the following equality holds*

$$\int_M \theta \psi \, d\text{vol}_g(x) + \int_0^T \int_M \theta \Lambda_g \psi - \theta u \cdot \nabla_g \psi \, d\text{vol}_g(x) \, dt = \int_M \theta_0 \psi \, d\text{vol}_g(x),$$

for each test function $\psi \in C^\infty([0, T] \times M)$.

Proof of Theorem A.1. We will prove it using Galerkin approximations. For each integer $m > 0$, consider the truncations θ_m given by

$$\theta_m(x, t) = \sum_{k=0}^m f_k(t) Y_k(x),$$

where Y_k are the eigenfunctions of $-\Delta_g$ with eigenvalues λ_k^2 . Denote by \mathbb{P}_m the projection onto the space generated by $\{Y_k\}_{k=1}^m$. Then for each fixed m , let us consider the truncated system:

$$\begin{cases} \partial_t \theta_m + \mathbb{P}_m(u_m \cdot \nabla_g \theta_m) + \Lambda_g \theta_m = 0, \\ u_m = \Psi[\theta_m], \\ \theta_m(x, 0) = \mathbb{P}_m \theta_0(x). \end{cases}$$

Although the above system seems to be a partial differential equation, it can be interpreted as a system of ordinary differential equations for the coefficients, $f_k(t)$,

$$f'_k(t) = -\lambda_k f_k(t) + \sum_{\ell=1}^m a_{k,\ell}(t) f_\ell(t),$$

where $a_{k,\ell}(t) = \int_M u(x, t) \cdot \nabla_g Y_k(x) Y_\ell(x) d\text{vol}_g(x)$ and the initial condition is provided by $\theta_m(0) = \mathbb{P}_m(\theta_0)$. The initial condition has bounded energy which, taking into account the nature of Ψ , implies the velocity is bounded in L^2 . From this it is easily proved that the coefficients $a_{k,\ell}$ are uniformly bounded. This allows us to use standard Picard-Lindelöf existence and uniqueness theorem for ordinary differential equations to show global existence of $f_k(t)$ (cf. [CL55], p. 20). Moreover, thanks to the divergence free condition the nonlinear term vanishes after integration by parts

$$\int_M \mathbb{P}_m(u_m \cdot \nabla_g \theta_m) \theta_m d\text{vol}_g(x) = 0,$$

and we get the uniform energy estimate

$$\|\theta_m(T)\|_{L^2(M)}^2 + \int_0^T \|\theta_m(\tau)\|_{H^{1/2}(M)}^2 d\tau \leq \|\theta_0\|_{L^2(M)}^2.$$

This implies θ_m is uniformly bounded in $L^\infty(0, T; L^2(M)) \cap L^2(0, T; H^{1/2}(M))$ for every $T > 0$. This ensures that, up to subsequence, θ_m converges weakly to some $\theta \in L^2(0, T; H^{1/2}(M))$ the limit still belongs to $L^\infty(0, T; L^2(M))$. We may force the subsequence to be such that $\theta_m(T)$ converges weakly to $\theta(T)$ in $L^2(M)$. In what remains we will show that the limit function θ obtained above is a weak solution of the Cauchy initial problem as stated.

Testing the truncated equation against some $\psi \in C^\infty([0, T] \times M)$ we obtain

$$\begin{aligned} \int_M \theta_m(x, T) \psi d\text{vol}_g(x) + \int_0^T \int_M \theta_m \Lambda \psi d\text{vol}_g(x) \\ - \int_0^T \int_M (\theta_m u_m) \cdot \nabla_g \psi d\text{vol}_g(x) dt = \int_M P_m \theta_0 \psi d\text{vol}_g(x). \end{aligned}$$

If we can take limits inside the integrals the weak formulation would be satisfied and we would be done. Since $\theta_m(T)$ converges weakly to $\theta(T)$ in L^2 it follows that

$$\int_M \theta_m(x, T) \psi d\text{vol}_g(x) \rightarrow \int_M \theta(x, T) \psi d\text{vol}_g(x),$$

while since $\|\mathbb{P}_m f - f\|_{L^2(M)} \rightarrow 0$ as we tend $m \rightarrow \infty$ for any $f \in L^2(M)$ it follows that

$$\int_M \mathbb{P}_m \theta_0(x) \psi d\text{vol}_g(x) \rightarrow \int_M \theta_0(x) \psi d\text{vol}_g(x).$$

The convergence of the second term above follows from the weak convergence in $L^2(0, T; H^{1/2}(M))$, indeed

$$\int_0^T \int_M \theta_m \Lambda \psi d\text{vol}_g(x) \rightarrow \int_0^T \int_M \theta \Lambda \psi d\text{vol}_g(x),$$

since $\Lambda\psi$ is still a test function. Regarding the nonlinear term, we realize that the weak convergence is not sufficient. We need some stronger convergence. It would be enough to prove that

$$\theta_m \rightarrow \theta \text{ strongly in } L^2(0, T; L^2(M)).$$

Let us show how this actually helps to deal with the nonlinear term. Indeed, for any test $\psi \in C^\infty([0, T] \times M)$ we would have

$$\begin{aligned} \left| \int_0^T \int_M \theta_m u_m \cdot \nabla_g \psi - \theta u \cdot \nabla_g \psi d\text{vol}_g(x) dt \right| &\leq \left| \int_0^T \int_M (\theta_m - \theta) u_m \cdot \nabla_g \psi d\text{vol}_g(x) dt \right| \\ &\quad + \left| \int_0^T \int_M \theta (u_m - u) \cdot \nabla_g \psi d\text{vol}_g(x) dt \right|, \end{aligned}$$

which can be bounded by

$$\begin{aligned} C(\psi) \left\{ \|\theta_m - \theta\|_{L^2(0, T; L^2(M))} \|u_m\|_{L^2(0, T; L^2(M))} \right. \\ \left. + \|u_m - u\|_{L^2(0, T; L^2(M))} \|\theta\|_{L^2(0, T; L^2(M))} \right\}, \end{aligned}$$

and now it becomes obvious that it converges to zero as $m \rightarrow \infty$.

To prove the claimed strong convergence we will proceed through the Aubin-Lions compactness lemma. To do so let us first observe that

$$\partial_t \theta_m \text{ is bounded in } L^{(n+1)/n}(0, T; B),$$

where $B = W^{-1, \frac{n+1}{n}}(M) + H^{-1/2}(M)$. Indeed the energy estimate above implies that

$$\theta_m \text{ is uniformly bounded in } L^{2(n+1)/n}(0, T; L^{2(n+1)/n}(M)).$$

As before, recall Ψ is a zero order operator it conserves $L^p(M)$ norms for $p \in (1, \infty)$, it follows then that

$$u_m \text{ is also bounded in } L^{2(n+1)/n}(0, T; L^{2(n+1)/n}(M)).$$

Therefore, $\nabla_g \cdot (u_m \theta_m)$ is bounded in $L^{(n+1)/n}(0, T; W^{-1, (n+1)/n}(M))$. Lastly, $\Lambda \theta_m$ is bounded in $L^2(0, T; H^{-1/2}(M))$. From the equation itself we get that $\partial_t \theta_m$ is bounded in $L^{(n+1)/n}(0, T; B)$. Now $H^{1/2}(M) \rightarrow L^2(M)$ is a compact inclusion by Rellich's theorem while by Sobolev $L^2(M)$ is continuously embedded into B . As a consequence the Aubin-Lions lemma provides the strong convergence (cf. [Lio69]). Finally, notice that $T > 0$ was arbitrary, proving that θ is a global weak solution. \square

Local existence, conditional global regularity and weak-smooth uniqueness

Here we will provide a sketchy idea of the proof dealing with the local existence of solutions to the critical SQG on a two dimensional orientable compact manifold (in particular \mathbb{S}^2), as well as the conditional global regularity and the weak-smooth uniqueness. The proofs follow by well-known techniques based on energy estimates which we include deliberately in this appendix for completeness.

B.1 Local existence in $H^s(M)$ for $s > \frac{3}{2}$

To start, we have that

$$\frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2(M)}^2 = -(u \cdot \nabla_g \theta)_{L^2(M)} - (\Lambda_g \theta, \theta)_{L^2(M)},$$

Using the divergence free condition $\operatorname{div}_g u = 0$ and the self-adjointness of the fractional laplacian, we have that

$$\|\theta(t)\|_{L^2(M)}^2 \leq \|\theta_0\|_{L^2(M)}^2,$$

implying the decay of the L^2 -norm. Next, to estimate higher-order Sobolev norms, recall that

$$\Lambda = (-\Delta_g)^{\frac{1}{2}}, \quad \|f\|_{H^s(M)} = \|\Lambda^s \theta\|_{L^2(M)}.$$

Then, computing the evolution of the H^s norm,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^s \theta\|_{L^2(M)}^2 &= -(\Lambda^s(u \cdot \nabla_g \theta), \Lambda^s \theta)_{L^2(M)} - \|\Lambda^{1+s} \theta\|_{L^2(M)}^2 \\ &= -(u \cdot \nabla_g \Lambda^s \theta, \Lambda^s \theta)_{L^2(M)} + ([\Lambda^s, u \cdot \nabla_g] \theta, \Lambda^s \theta)_{L^2(M)} - \|\Lambda^{\frac{1}{2}+s} \theta\|_{L^2(M)}^2. \end{aligned}$$

We rewrote the nonlinear term to use the divergence free condition, but we need to compute the commutator $[\Lambda^s, \nabla_g]$. It is well-known that using the Kato-Ponce commutator (cf. [KP88], and later extended for manifolds [Tay81]) we have that,

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^2(M)} \leq C \left(\|f\|_{C^1(M)} \|g\|_{H^{s-1}(M)} + \|f\|_{H^s(M)} \|g\|_{L^\infty(M)} \right),$$

for $s > 0$. Therefore, if we denote by $P := \nabla_g$ which is a first order differential operator, we have that

$$[\Lambda^s, u \cdot \nabla_g] \theta = \Lambda^s(uP\theta) - uP(\Lambda_g^s \theta) + u[\Lambda, P]\theta.$$

Hence,

$$\begin{aligned} \|[\Lambda^s, u \cdot \nabla_g] \theta\|_{L^2(M)} &\leq \| \Lambda^s(uP\theta) - uP(\Lambda_g^s \theta) \|_{L^2(M)} + \|u[\Lambda_g, P]\theta\|_{L^2(M)} \\ &\leq C \left(\|u\|_{C^1(M)} \|\theta\|_{H^s(M)} + \|u\|_{H^s(M)} \|\theta\|_{C^1(M)} \right) + C \|u\|_{L^\infty(M)} \|\theta\|_{H^s(M)} \end{aligned}$$

where we have used the Kato-Ponce commutator for the first term and the fact that commutator $[\Lambda^s, P] \in OPS^s(M)$. We conclude,

$$\frac{d}{dt} \|\Lambda^s \theta\|_{L^2(M)}^2 \leq C \|\Lambda^s \theta\|_{L^2(M)}^2 \|\Lambda^{2+\epsilon} \theta\|_{L^2(M)} - \|\Lambda^{\frac{1}{2}+s} \theta\|_{L^2(M)}^2,$$

for every $\epsilon > 0$, since $\|u\|_{C^1(M)} \leq \|\Lambda^{2+\epsilon} \theta\|_{L^2(M)}$. Therefore, choosing $\epsilon = s - \frac{3}{2}$ we get

$$\frac{d}{dt} \|\Lambda^s \theta\|_{L^2(M)}^2 \leq C \|\Lambda^s \theta\|_{L^2(M)}^4,$$

which yields the desired a priori estimate for every $s > \frac{3}{2}$.

Remark B.1. Once the local a priori bound is established, the construction of a local solution can be obtained through standard procedure. For instance, one can define mollifiers $J_\epsilon = \phi(\epsilon \Delta_g)$, ϕ is a real valued and $C^\infty(\mathbb{R})$ with compact support. Once, the equation is mollified, the short time existence for $\epsilon > 0$ is trivial, since is a finite system of ODE's. The main point, is to obtain independent a priori estimates on ϵ , to pass to the limit (which follow also by the above estimates). We shall omit the construction part to avoid redundancy.

Remark B.2. Notice that during the time of existence T , the equation implies that

$$\int_0^T \int_M |\Lambda^{s+\frac{1}{2}} \theta|^2 < \infty.$$

As a consequence we can replace our initial time by some t_0 as close as we want to the initial time so that $\theta(x, t_0) \in H^{2+\epsilon}$ for some $\epsilon > 0$. Sobolev embedding implies now that $\nabla \theta(\cdot, t_0) \in L^\infty$ which is all we need.

B.2 Conditional global existence in $H^s(M)$

In this section we show how global control of $\|\nabla_g \theta\|_{L^\infty}$ provides immediately global well posedness (cf. Theorem 1.3) in higher order Sobolev spaces $H^s(M)$ for $s \in \mathbb{N}$. We do employ the critical dissipation and since the case $s = 0, 1$ are easier to handle directly we will provide a sketch with $s > 1$.

Consider the evolution of the H^s norm of the solution,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\theta\|_{H^s(M)}^2 + \|\theta\|_{H^{s+\frac{1}{2}}(M)}^2 &= - \sum_{s=\alpha} (D^\alpha(u \cdot \nabla_g \theta), D^\alpha \theta)_{L^2(M)} \\
&= - \sum_{s=\alpha} \sum_{|\beta| \leq s} (c_{\alpha,\beta}(D^\beta u)(D^{\alpha-\beta} \nabla_g \theta), D^\alpha \theta)_{L^2(M)} \\
&\leq C \sum_{j=1}^s ((D^j u)(D^{s-j} \nabla_g \theta), D^s \theta)_{L^2(M)} + (u \cdot \nabla_g D^s \theta, D^s \theta)_{L^2(M)} \\
&\quad + (u \cdot [D^s, \nabla_g] \theta, D^s \theta)_{L^2(M)} := I_1 + I_2 + I_3,
\end{aligned}$$

where we have set $|\beta| = j$ and $|\alpha - \beta| = s - j$. The second term I_2 is zero using the divergence free condition. Let us estimate the term I_1 and for the sake of simplicity we treat the case $j = 1$ indicating later how to proceed in the general case. Using the Hölder inequality and the boundedness of the Riesz transform in $L^p(M)$ spaces for $1 < p < \infty$, yield

$$\begin{aligned}
|I_1| &\leq C \|Du\|_{L^3(M)} \|D^s \theta\|_{L^3(M)}^2 \\
&\leq \|D\theta\|_{L^3(M)} \|D^s \theta\|_{L^3(M)}^2.
\end{aligned}$$

Again Hölder and fractional Sobolev inequality imply

$$\begin{aligned}
\|D^s \theta\|_{L^3(M)}^2 &\leq \left(\int_M |D^s \theta|^2 d\text{vol}_g(x) \right)^{\frac{1}{3}} \left(\int_M |D^s \theta|^4 d\text{vol}_g(x) \right)^{\frac{1}{3}} \\
&\leq \|D^s \theta\|_{L^2(M)}^{\frac{2}{3}} \|D^s \theta\|_{L^4(M)}^{\frac{4}{3}} \\
&\leq \|D^s \theta\|_{L^2(M)}^{\frac{2}{3}} \|D^{s+\frac{1}{2}} \theta\|_{L^2(M)}^{\frac{4}{3}},
\end{aligned}$$

and Gagliardo-Nirenberg's inequality yields

$$\|D^s \theta\|_{L^2(M)} \leq C \|D\theta\|_{L^2(M)}^{\frac{1}{2s-1}} \|D^{s+1/2} \theta\|_{L^2(M)}^{1-\frac{1}{2s-1}}.$$

Therefore, we have that

$$\begin{aligned}
|I_1| &\leq C \|D\theta\|_{L^3(M)} \|D\theta\|_{L^2(M)}^{\frac{2}{3(2s-1)}} \|D^{s+\frac{1}{2}} \theta\|_{L^2(M)}^{\frac{4}{3} + (1-\frac{1}{2s-1})\frac{2}{3}} \\
&\leq C \|D\theta\|_{L^3(M)} \|D\theta\|_{L^2(M)}^{\frac{2}{3(2s-1)}} \|D^{s+\frac{1}{2}} \theta\|_{L^2(M)}^{2-\frac{2}{3(2s-1)}}.
\end{aligned}$$

Now, it is easy to see that the term I_3 is easier to bound since,

$$|I_3| \leq C \|u\|_{L^3(M)} \|D^s \theta\|_{L^3(M)}^2 \leq C \|D^s \theta\|_{L^3(M)}^2,$$

and $u \in L^3$ is uniformly bounded. Then appropriate Hölder and Sobolev inequalities dispose of this case as above. In conclusion, since we have $\|\nabla_g \theta\|_{L^\infty(M)}$ uniformly bounded and M is a compact manifold, the above gives

$$\frac{1}{2} \frac{d}{dt} \|\theta\|_{H^s(M)}^2 + \|\theta\|_{H^{s+\frac{1}{2}}(M)}^2 \leq C \|\theta\|_{H^{s+\frac{1}{2}}(M)}^{2-\frac{2}{3(2s-1)}}.$$

Finally, one may use Young's inequality to subsume in the dissipation term the right hand side. As a consequence

$$\frac{d}{dt} \|\theta(t)\|_{H^s}^2 \leq C,$$

where the $C = C(s, M, \|\nabla_g \theta\|_{L^\infty})$, for every $t > 0$.

For the general case, $j > 1$, the same strategy would work taking into account the following Gagliardo-Nirenberg inequality

$$\|D^s \theta\|_{L^4(M)} \leq \|D^s \theta\|_{L^2(M)}^{\frac{1}{2}} \|D^{s+1} \theta\|_{L^2(M)}^{\frac{1}{2}}.$$

Hence the only case where we can not apply directly this estimation is when we have two terms with all the derivatives, which just occurs if $j = 1$.

B.3 Weak-smooth uniqueness of solutions

In this section, we will prove the smooth-weak uniqueness of solutions, this is any two weak solutions θ_1 and θ_2 with same initial data coincide provided that one of them is smooth. We denote the corresponding velocity vectors by u_1 and u_2 , respectively. Let us define $\tilde{\theta} = \theta_1 - \theta_2$ and suppose without loss of generality that θ_2 is smooth. Then the evolution of the L^2 norm of $\tilde{\theta}$ can be estimated as follows

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{\theta}\|_{L^2(M)}^2 + \|\tilde{\theta}\|_{H^{1/2}(M)}^2 &= - \int_M u_1 \cdot \nabla_g(\tilde{\theta}^2) \, d\text{vol}_g - \int_M (u_1 - u_2) \cdot \nabla_g \theta_2 \tilde{\theta} \, d\text{vol}_g \\ &\leq C \|u_1 - u_2\|_{L^2(M)} \|\nabla_g \theta_2\|_{L^\infty(M)} \|\tilde{\theta}\|_{L^2(M)} \\ &\leq C \|\nabla_g \theta_2\|_{L^\infty} \|\tilde{\theta}\|_{L^2(M)}^2 \\ &\leq C \|\tilde{\theta}\|_{L^2(M)}^2, \end{aligned}$$

where we have used that the first term in the right hand side vanishes due to the incompressibility of u_1 , while the second can be bounded using the boundedness of the Riesz transforms and the hypothesis of θ_2 being smooth for all times. Invoking Gönwall's inequality yields that $\|\tilde{\theta}\|_{L^2(M)} = 0$ which implies that $\theta_1 = \theta_2$, and hence uniqueness.

Other results

In this appendix, we present other results that I have been done during my PhD. The first two are enclosed in the area of stochastic partial differential equations and the other one deals with qualitative properties of the heat kernel. More precisely, in Section C.1 we study the well-posedness of two stochastic equations arising from fluid mechanics [AOL18a; AO+18e]. In Section C.2, we show a monotonicity property of the fundamental solution of the heat equation, namely, the heat kernel [AO+18d].

C.1 Stochastic transport: the Boussinesq system and Burgers' equation

Stochastic Partial Differential Equations (SPDEs) serve as fundamental models of physical systems subject to random interactions or inputs. In particular, stochasticity is a powerful mechanism to understand the problem of turbulence in fluid dynamics. Most deterministic models are intractable and can not be solved accurately. However, statistical averages and properties of the solution are typically more robust and are incredible useful to deal with small scales (i.e. called in the literature, stochastic parametrisation). Uncertainty due to several phenomena is considered a drastic problem in weather prediction and climate modeling. Stochasticity can help account for this kind of uncertainty.

Recently, D. Holm derived stochastic PDE's for fluid dynamics arising stochastic variational principles [Hol15]. The stochasticity is added in such a way that several geometric quantities in the Euler-Poincaré formulation are preserved.

Last year, D. Crisan, F. Flandoli and D. Holm investigated the local existence, uniqueness and possible singularities of the 3D Euler equation driven by stochastic transport, recovering essentially the same solution properties known for the deterministic 3D Euler equation [Cri+18]. This paper got my attention, and I started thinking about how this stochastic transport affects other relevant equation in fluid dynamics.

C.1.1 The Boussinesq equation with multiplicative cylindrical noise

The Boussinesq equations are widely considered as a fundamental model for the study of large scale atmospheric and oceanic flows, built environment, dispersion of dense gases, and internal dynamical structure of stars.

In collaboration with A. Bethencourt de León we introduced stochasticity into the incompressible 2D Boussinesq equations using the approach of [Hol15]. The new stochastic equations are given by

$$d\omega + \mathcal{L}_u \omega dt + \sum_{i=1}^{\infty} \mathcal{L}_{\xi_i} \omega \circ dB_t^i = \partial_x \theta dt, \quad (\text{C.1})$$

$$d\theta + \mathcal{L}_u \theta dt + \sum_{i=1}^{\infty} \mathcal{L}_{\xi_i} \theta \circ dB_t^i = 0, \quad (\text{C.2})$$

where $\{B_t^i\}_{i \in \mathbb{N}}$ is a countable set of independent Brownian motions, $\{\xi_i(\cdot)\}_{i \in \mathbb{N}}$ is a countable set of prescribed functions depending only on the spatial variable, and \circ means that the stochastic integral is interpreted in the Stratonovich sense. We prove the local existence of regular solutions and construct a blow-up criterion, [AOL18a].

The idea of the proof hinges on proving a new commutator estimate to deal with the high order terms appearing due to the stochastic transport to close the energy estimates. Moreover, we derive general Lie derivatives estimates which are of independent interest, which can be applied to study broader and much more general noise type terms. Several applications and comments about this result will be part of a short communication note which is still in preparation.

C.1.2 The Burgers' equation with stochastic transport

In a similar research direction, in a joint work with A. Bethancourt de León and S. Takao [AO+18e], we studied the well-posedness of a stochastic Burgers' equation

$$du(t, x) + u(t, x) \partial_x u(t, x) dt + \sum_{k=1}^{\infty} \xi_k(x) \partial_x u(t, x) \circ dB_t^k = \nu \partial_{xx} u(t, x) dt. \quad (\text{C.3})$$

Compared with the well-studied Burgers' equation with additive noise (i.e. noise appears as an external random force), this type of noise arises by taking the diffusive limit of the Lagrangian flow map regarded as a composition of a slow mean flow and a rapidly fluctuating one. We are interested in investigating how the stochastic transport affects the Burgers' equation, and in particular whether this noise could prevent the system from developing shocks. We prove the following results:

1. For $\nu = 0$, equation (C.3) has a unique solution of class H^s for $s > 3/2$ until some stopping time $\tau > 0$.
2. However, shock formation cannot be avoided a.s. in the case $\xi(x) = \alpha x + \beta$ and for a broader class of $\{\xi_k(\cdot)\}_{k \in \mathbb{N}}$, we can prove that it occurs in expectation.
3. For $\nu > 0$, we have global existence and uniqueness in H^2 .

C.2 Monotonicity property of the heat kernel

The heat equation is one of the quintessentials among mathematical models for physical phenomena. Over the years, several properties of this equation had been studied from different points of view, including for instance: probabilistic, geometric and physical. We will focus on the fundamental solution of the heat equation, namely, the heat kernel. For small times a parametrix is well known.

In a joint work with F. Chamizo, A.D. Martínez and A. Mas [AO+18d], we give a rigorous proof of the following intuitively true result and some generalizations to manifolds with symmetries.

Theorem C.1. *Let M be \mathbb{R}^n , \mathbb{S}^n or \mathbb{H}^n . Then, for any fixed $x \in M$ and time $t \in (0, \infty)$, the heat kernel $G(x, y, t)$ is a decreasing function of the geodesic distance $d(x, y)$.*

This is known due to explicit formulae for the euclidean and hyperbolic spaces. For the sphere the best results available in the literature deal with the one, two and three dimensional cases [And13]. The proof is quite elaborated, based on specific estimates using spherical harmonics and does not seem to generalize in a straightforward way to higher dimensions. Our neat proof, nevertheless, is built in a delicate application of the parabolic maximum principle. The same arguments also apply to more general situations described below, of which Theorem C.1 is a rather beautiful particular case.

Theorem C.2. *Let $M \subseteq \mathbb{R}^n$ be a smooth, compact and connected hypersurface of revolution around the x_n axis. If x is a point of intersection of M and the x_n axis, then the associated heat kernel $G(x, y, t)$ decreases as a function of the geodesic distance $d(x, y)$ for any fixed $t > 0$.*

The same proof covers the noncompact situation even in an intrinsic geometric setting beyond hypersurfaces of \mathbb{R}^n .

We also tackle the Dirichlet and Neumann heat kernels, G_D and G_N respectively, of a smooth hypersurface of revolution $M \subseteq \mathbb{R}^n$ with boundary.

Theorem C.3. *Let $M \subseteq \mathbb{R}^n$ be smooth and connected hypersurface of revolution around the x_n axis with boundary $\partial M \neq \emptyset$.*

- (i) *If x is a point of intersection of the relative interior of M and the x_n axis, then the associated heat kernel with Dirichlet boundary condition $G_D(x, y, t)$ decreases as a function of the geodesic distance $d(x, y)$ for any fixed time $t > 0$.*
- (ii) *If x is a point of intersection of the relative interior of M and the x_n axis, then the associated heat kernel with Neumann boundary condition $G_N(x, y, t)$ decreases as a function of the geodesic distance $d(x, y)$ for any fixed time $t > 0$.*

We conclude by applying the above to exhibit implications to pointwise inequalities for orthogonal polynomials and the kernel of the fractional Laplace-Beltrami operator on the sphere.

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